MATH301

Contents

Chapter 1

VECTOR FIELD

Definition

A vector is a quantity having both magnitude and direction.

1.1 Product of vectors

we have two types of product namely dot and vector product.
Dot product of two vectors is a scalar, while vector product is a vector. eg for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$

1.2 Vector Differentiation

1.2.1 Ordinary Derivative of vectors

Let $R(t)$ be a vector depending on a single scalar t

$$\frac{\Delta R}{\Delta t} = \frac{R(t + \Delta t) - R(t)}{\Delta t}$$

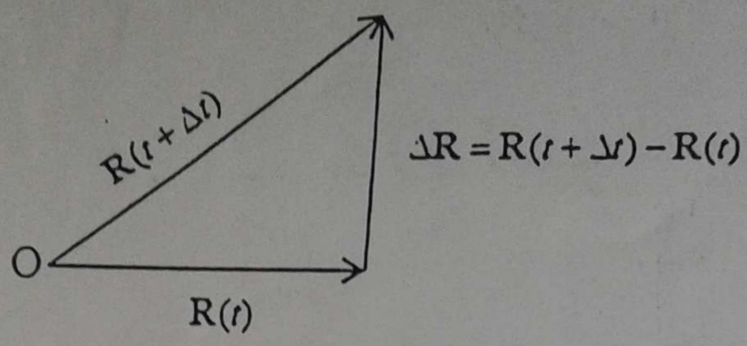

Figure 1.1:

where $\triangle t$ denotes an increment in t see figure 1.1 below The ordinary derivative of the vector $R(t)$ with respect to t is given by

$$\frac{dR}{dt} = \lim_{\Delta t\to 0} \frac{\Delta R}{\Delta t} = \lim_{\Delta t\to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t}$$

if it exist.

1.2.2 Space Curves

If in particular $R(t)$ is the position vector $r(t)$ joining O of a coordinate system and any point (x, y, z) then

$$\mathrm{r}(t) = x(t)\mathrm{i} + y(t)\mathrm{j} + z(t)\mathrm{k}$$

and specification of the vector function $\mathrm{r}(t)$ defines x, y and z as functions of t.
As t changes, the terminal point of r describe a space curve having parametric equations $x(t), Y(t), Z(t)$ then

$$\frac{\Delta \mathrm{r}}{\Delta t} = \frac{\mathrm{r}(t+\Delta t) - \mathrm{r}(t)}{\Delta t}$$

is a vector in the direction of $\Delta \mathrm{r}$ see the figure 1.2 below

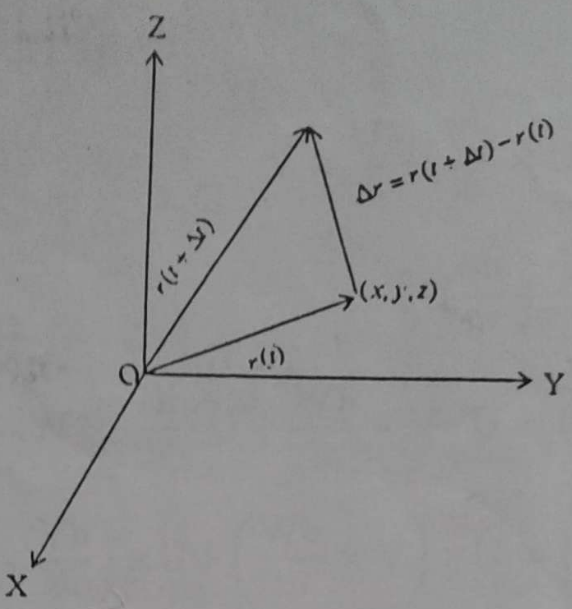

Figure 1.2:

If $\lim_{\Delta t} \to 0 \frac{\Delta \mathrm{r}}{\Delta t} = \frac{d\mathrm{r}}{dt}$ exist. The limit will be a vector, in the direction of the tangent to the space curve at (x, y, z) and its giving by

$$\frac{d\mathrm{r}}{dt} = \frac{dx}{dt}\mathrm{i} + \frac{dy}{dt}\mathrm{j} + \frac{dz}{dt}\mathrm{k}$$

If $t$is the time, $\frac{d\mathrm{r}}{dt}$ represent velocity **V** with which the terminal point of **r** describe the curve. Similarly $\frac{d\mathrm{V}}{dt} = \frac{d^2\mathrm{r}}{dt^2}$ represent the acceleration **a** along the curve.

1.2.3 Differentiation formulae

If **U**, **V** and **W** are differentiable functions of a scalar t and ϕ is a differentiable scalar function of t then

1. $\frac{d}{dt}(\mathrm{U} + \mathrm{V}) = \frac{d\mathrm{U}}{dt} + \frac{d\mathrm{V}}{dt}$

2. $\frac{d}{dt}(\mathrm{U} \cdot \mathrm{V}) = \mathrm{U} \cdot \frac{d\mathrm{V}}{dt} + \frac{d\mathrm{U}}{dt} \cdot \mathrm{V}$

3. $\frac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{U}\times\frac{d\mathbf{V}}{dt} + \frac{d\mathbf{U}}{dt}\times\mathbf{V}$

4. $\frac{d}{dt}(\phi\mathbf{U}) = \phi\frac{d\mathbf{U}}{dt} + \frac{d\phi}{dt}\mathbf{U}$

5. $\frac{d}{dt}(\mathbf{U}\cdot\mathbf{V}\times\mathbf{W}) = \mathbf{U}\cdot\mathbf{V}\times\frac{d\mathbf{W}}{dt} + \mathbf{U}\cdot\frac{d\mathbf{V}}{dt}\times\mathbf{W} + \frac{d\mathbf{U}}{dt}\cdot\mathbf{V}\times\mathbf{W}$

6. $\frac{d}{dt}(\mathbf{U}\times\mathbf{V}\times\mathbf{W}) = \mathbf{U}\times\left(\mathbf{V}\times\frac{d\mathbf{W}}{dt}\right) + \mathbf{U}\times\left(\frac{d\mathbf{V}}{dt}\times\mathbf{W}\right) + \frac{d\mathbf{U}}{dt}\times(\mathbf{V}\times\mathbf{W})$

1.2.4 Partial Differentiation of Vectors

If $\mathbf{F}$ is a vector depending on more than one scalar variable say x, y, z for example $\mathbf{F} = F(x, y, z)$ the partial derivatives of $\mathbf{F}$ with respect to the variables x, y and z are given respectively as

$$\frac{\partial \mathbf{F}}{\partial x} = \lim_{\Delta x\to 0}\frac{\mathbf{F}(x+\Delta x, y, z) - \mathbf{F}(x, y, z)}{\Delta x}$$

$$\frac{\partial \mathbf{F}}{\partial y} = \lim_{\Delta y\to 0}\frac{\mathbf{F}(x, y+\Delta y, z) - \mathbf{F}(x, y, z)}{\Delta y}$$

$$\frac{\partial \mathbf{F}}{\partial z} = \lim_{\Delta z\to 0}\frac{\mathbf{F}(x, y, z+\Delta z) - \mathbf{F}(x, y, z)}{\Delta z}$$

if the limits exist.

1.2.5 Higher Derivative

$$\frac{\partial^2 \mathbf{F}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial x}\right), \frac{\partial^2 \mathbf{F}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{F}}{\partial y}\right), \frac{\partial^2 \mathbf{F}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial y}\right)$$

$$\frac{\partial^2 \mathbf{F}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{F}}{\partial x}\right), \frac{\partial^3 \mathbf{F}}{\partial x \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial z^2}\right)$$

and so on.

If $\mathbf{F}$ has continuous partial derivatives of the second order atleast then $\frac{\partial^2 \mathbf{F}}{\partial x \partial y} = \frac{\partial^2 \mathbf{F}}{\partial y \partial x}$ that is, the order of differentiation does not matter.

1:2.6 Rules of Partial Differentiation

1. $\dfrac{\partial}{\partial x}(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\cdot\dfrac{\partial \mathbf{G}}{\partial x} + \dfrac{\partial \mathbf{F}}{\partial x}\cdot\mathbf{G}$

2. $\dfrac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\times\dfrac{\partial \mathbf{G}}{\partial x} + \dfrac{\partial \mathbf{F}}{\partial x}\times\mathbf{G}$

3. $\dfrac{\partial^2}{\partial y\partial x}(\mathbf{F}\cdot\mathbf{G}) = \dfrac{\partial}{\partial t}\left(\dfrac{\partial}{\partial x}(\mathbf{F}\cdot\mathbf{G})\right) = \dfrac{\partial}{\partial x}\left(\mathbf{F}\cdot\dfrac{\partial \mathbf{G}}{\partial x} + \dfrac{\partial \mathbf{F}}{\partial x}\cdot\mathbf{G}\right)$
$= \mathbf{F}\cdot\dfrac{\partial^2 \mathbf{G}}{\partial y\partial x} + \dfrac{\partial \mathbf{F}}{\partial y}\cdot\dfrac{\partial \mathbf{G}}{\partial x} + \dfrac{\partial \mathbf{F}}{\partial x}\cdot\dfrac{\partial \mathbf{G}}{\partial y} + \dfrac{\partial^2 \mathbf{F}}{\partial y\partial x}\cdot\mathbf{G}$ e.t.c.

1.2.7 Differential of Vectors

Follows the rules similar to those of elementary calculus, for example

1. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ then
$d\mathbf{F} = dF_1\mathbf{i} + dF_2\mathbf{j} + dF_3\mathbf{k}$

2. $d(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\cdot d\mathbf{G} + d\mathbf{F}\cdot\mathbf{G}$

3. $d(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\times d\mathbf{G} + d\mathbf{F}\times\mathbf{G}$

4. If $F = F(x, y, z)$ then $dF = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy + \dfrac{\partial F}{\partial z}dz$ e.t.c.

Exercise 1.1

1. If $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ where x, y, z are differentiable functions of the scalar t. Prove that

$$\frac{d\mathbf{R}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

2. Let A depend on x, y, z, t where x, y, z depend on t. Prove that

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{A}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{A}}{\partial z}\frac{dz}{dt}$$

1.3 Gradient, Divergence and Curl

The vector differential operator Del, written as ∇ is define by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$
$$= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

1.3.1 The Gradient

Let $\phi(x, y, z)$ be define and differentiable at each point (x, y, z) in a certain region of space(ie ϕ define a differentiable scalar field) then the gradient of ϕ, written as $\nabla\phi$ or gradϕ is defined by

$$\nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ define a vector field.
The component of $\nabla\phi$ in the direction of a unit vector $\mathbf{a}$ is given by $\nabla\phi \cdot \mathbf{a}$ and is called the directional derivative of ϕ in the direction of $\mathbf{a}$. Physically this is the rate of change of ϕ at (x, y, z) in the direction of $\mathbf{a}$.

1.3.2 The Divergence

Let $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be define and differentiable at each point (x, y, z) in a region at space(i.e $\mathbf{V}$ defines a differentiable vector field) then the divergence of $\mathbf{V}$, written as $\nabla \cdot \mathbf{V}$ or div $\mathbf{V}$ is define by

$$\nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note $\nabla \cdot \mathbf{V} \neq \mathbf{V} \cdot \nabla$.

1.3.3 The Curl

Let $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ be define and differentiable at each point (x, y, z) in a region at space(i.e $\mathbf{V}$ defines a differentiable vector field) then the curl or rotation of $\mathbf{V}$, written as $\nabla \times \mathbf{V}$ or Curl$\mathbf{V}$ is define by

$$\nabla \times \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

Formulae involving ∇ if A and B are differentiable vector functions ϕ and ψ are differentiable scalar functions of position (x, y, z) then

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$ or $grad(\phi + \psi) = grad\phi + grad\psi$
2. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$ or $div(\mathbf{A} + \mathbf{B}) = div\mathbf{A} + div\mathbf{B}$
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ or $curl(\mathbf{A} + \mathbf{B}) = curl\mathbf{A} + curl\mathbf{B}$
4. $\nabla \cdot (\phi\mathbf{A}) = \nabla\phi \cdot \mathbf{A} + \phi\nabla \cdot \mathbf{A}$
5. $\nabla \times (\phi\mathbf{A}) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$
6. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
7. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$
8. $\nabla \cdot (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$
9. $\nabla \cdot (\nabla\phi) = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$
   where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the Laplacian operator.
10. $\nabla \times (\nabla\phi) = 0$. The curl of the gradient ϕ is zero or $curl(grad\phi) = 0$
11. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. The divergence of curl A is zero. or $div(curl\mathbf{A}) = 0$.
12. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

Example 1.1

Find $\nabla\phi$ if $\phi(x, y, z) = 2xy^3 - 3xyz$. at the point $(1, -1, 1)$.

Solution

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathrm{i} + \frac{\partial}{\partial y}\mathrm{j} + \frac{\partial}{\partial z}\mathrm{k}\right)(2xy^3 - 3xyz)$$

$$= \mathrm{i}\frac{\partial}{\partial x}(2xy^3 - 3xyz) + \mathrm{j}\frac{\partial}{\partial y}(2xy^3 - 3xyz) + \mathrm{k}\frac{\partial}{\partial z}(2xy^3 - 3xyz)$$

$$= (2y^3 - 3yz)\mathrm{i} + (6xy^2 - 3xz)\mathrm{j} - 3xy\mathrm{k}$$

Diff with respect to x, y & z. substitute the point

at $(1, -1, 1)$

$$= (-2 + 3)\mathrm{i} + (+6 - 3)\mathrm{j} + 3k = \sout{\mathrm{i} - 9\mathrm{j} + 3k}$$

$\hat{i} + 3\hat{j} + 3K$

Example 1.2

Find $\nabla\phi$ if (a) $\phi = \ln|\mathrm{r}|$ (b) $\phi = \dfrac{1}{r}$ where $\mathrm{r} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$

Solution

(a) Since $\mathrm{r} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ then $|\mathrm{r}| = \sqrt{x^2 + y^2 + z^2}$ and

$\phi = \ln|\mathrm{r}| = \ln(x^2 + y^2 + z^2)^{1/2} = \dfrac{1}{2}\ln(x^2 + y^2 + z^2)$

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathrm{i} + \frac{\partial}{\partial y}\mathrm{j} + \frac{\partial}{\partial z}\mathrm{k}\right)\cdot\left(\frac{1}{2}\ln(x^2 + y^2 + z^2)\right)$$

$$= \frac{1}{2}\left[\mathrm{i}\frac{\partial}{\partial x}\left(\ln(x^2 + y^2 + z^2\right) + \mathrm{j}\frac{\partial}{\partial y}\left(\ln(x^2 + y^2 + z^2\right) + \mathrm{k}\frac{\partial}{\partial z}\left(\ln(x^2 + y^2 + z^2\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{2x}{x^2 + y^2 + z^2}\right)\mathrm{i} + \left(\frac{2y}{x^2 + y^2 + z^2}\right)\mathrm{j} + \left(\frac{2z}{x^2 + y^2 + z^2}\right)\mathrm{k}\right]$$

$$= \frac{x\mathrm{i} + y\mathrm{j} + z\mathrm{k}}{x^2 + y^2 + z^2} = \frac{\mathrm{r}}{r^2}$$

(b) $\phi = \dfrac{1}{r} = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-1/2}$

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathrm{i} + \frac{\partial}{\partial y}\mathrm{j} + \frac{\partial}{\partial z}\mathrm{k}\right)\cdot\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= \mathrm{i}\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-1/2} + \mathrm{j}\frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-1/2} + \mathrm{k}\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= \mathrm{i}\frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}2x + \mathrm{j}\frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}2y + \mathrm{k}\frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}2z$$

$$= \frac{-x\mathrm{i} - y\mathrm{j} - z\mathrm{k}}{(x^2 + y^2 + z^2)^{3/2}} = \frac{-\mathrm{r}}{r^3}$$

Example 1.3

Prove that (a) $\nabla\cdot(\mathrm{A}+\mathrm{B})=\nabla\cdot\mathrm{A}+\nabla\cdot\mathrm{B}$ (b) $\nabla\cdot(\phi\mathrm{A})=(\nabla\phi)\cdot\mathrm{A}+\phi(\nabla\cdot\mathrm{A})$

Solution

(a) Let $\mathbf{A}=A_1\mathrm{i}+A_2\mathrm{j}+A_3\mathrm{k}$ and $\mathrm{B}=B_1\mathrm{i}+B_2\mathrm{j}+B_3\mathrm{k}$ then

$$\nabla(\mathrm{A}+\mathrm{B})=\left(\mathrm{i}\frac{\partial}{\partial x}+\mathrm{j}\frac{\partial}{\partial y}+\mathrm{k}\frac{\partial}{\partial z}\right)((A_1+B_1)\mathrm{i}+(A_2+B_2)\mathrm{j}+(A_3+B_3)\mathrm{k})$$

$$=\frac{\partial}{\partial x}(A_1+B_1)+\frac{\partial}{\partial y}(A_2+B_2)+\frac{\partial}{\partial z}(A_3+B_3)$$

$$=\frac{\partial A_1}{\partial x}+\frac{\partial B_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial B_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial B_3}{\partial z}$$

$$=\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}+\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}$$

$$=\left(\mathrm{i}\frac{\partial}{\partial x}+\mathrm{j}\frac{\partial}{\partial y}+\mathrm{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathrm{i}+A_2\mathrm{j}+A_3\mathrm{k})+\left(\mathrm{i}\frac{\partial}{\partial x}+\mathrm{j}\frac{\partial}{\partial y}+\mathrm{k}\frac{\partial}{\partial z}\right)\cdot(B_1\mathrm{i}+B_2\mathrm{j}+B_3\mathrm{k})$$

$$=\nabla\cdot\mathrm{A}+\nabla\cdot\mathrm{B}$$

(b) Let $\mathbf{A}=A_1\mathrm{i}+A_2\mathrm{j}+A_3\mathrm{k}$ and $\phi(x,y,z)$ then

$$\nabla\cdot(\phi\mathrm{A})=\left(\mathrm{i}\frac{\partial}{\partial x}+\mathrm{j}\frac{\partial}{\partial y}+\mathrm{k}\frac{\partial}{\partial z}\right)\cdot(\phi A_1\mathrm{i}+\phi A_2\mathrm{j}+\phi A_3\mathrm{k})$$

$$=\frac{\partial}{\partial x}(\phi A_1)+\frac{\partial}{\partial y}(\phi A_2)+\frac{\partial}{\partial z}(\phi A_3)$$

$$=\frac{\partial\phi}{\partial x}A_1+\phi\frac{\partial A_1}{\partial x}+\frac{\partial\phi}{\partial y}A_2+\phi\frac{\partial A_2}{\partial y}+\frac{\partial\phi}{\partial z}A_3+\phi\frac{\partial A_3}{\partial z}$$

$$=\left(\frac{\partial\phi}{\partial x}A_1+\frac{\partial\phi}{\partial y}A_2+\frac{\partial\phi}{\partial z}A_3\right)+\left(\phi\frac{\partial A_1}{\partial x}+\phi\frac{\partial A_2}{\partial y}+\phi\frac{\partial A_3}{\partial z}\right)$$

$$=\left(\mathrm{i}\frac{\partial\phi}{\partial x}+\mathrm{j}\frac{\partial\phi}{\partial y}+\mathrm{k}\frac{\partial\phi}{\partial z}\right)\cdot(A_1\mathrm{i}+A_2\mathrm{j}+A_3\mathrm{k})+\phi\left(\mathrm{i}\frac{\partial}{\partial x}+\mathrm{j}\frac{\partial}{\partial y}+\mathrm{k}\frac{\partial}{\partial z}\right)\cdot(A_1\mathrm{i}+A_2\mathrm{j}+A_3\mathrm{k})$$

$$=(\nabla\phi)\cdot\mathrm{A}+\phi(\nabla\cdot\mathrm{A})$$

Example 1.4

Prove that $Curl(grad\phi)=0$ i.e $\nabla\times(\nabla\phi)=0$

Solution

Let $\phi(x, y, z)$ be a scalar field.

$$grad\phi = \left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \frac{\partial \phi}{\partial x}\mathrm{i} + \frac{\partial \phi}{\partial y}\mathrm{j} + \frac{\partial \phi}{\partial z}\mathrm{k}$$

$$Curl(grad\phi) = \left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right) \times \left(\frac{\partial \phi}{\partial x}\mathrm{i} + \frac{\partial \phi}{\partial y}\mathrm{j} + \frac{\partial \phi}{\partial z}\mathrm{k}\right)$$

$$= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \mathrm{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \mathrm{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + \mathrm{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) = 0$$

Example 1.5

Prove that $div(curl\mathrm{A}) = 0$ i.e $\nabla \cdot (\nabla \times \mathrm{A}) = 0$

Solution

Let $\mathrm{A} = A_1\mathrm{i} + A_2\mathrm{j} + A_3\mathrm{k}$

$$\nabla \times \mathrm{A} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \mathrm{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \mathrm{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \mathrm{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \mathrm{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\therefore = \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z}\right) - \left(\frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z}\right) + \left(\frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial y \partial x} + \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} = 0$$

Example 1.6

If $\mathrm{A} = xy^2\mathrm{i} + x^2yz\mathrm{j} + y^3z\mathrm{k}$, find (i) $\nabla \times \mathrm{A}$ (ii) $\nabla \cdot \mathrm{A}$ at $(-1, 1, -1)$

Solution

$$\text{(i) } \nabla \times \mathrm{A} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x^2yz & y^3z \end{vmatrix} = \mathrm{i}(3y^2z - x^2y) - \mathrm{j}(0 - 0) + \mathrm{k}(2xyz - 2xy)$$

at $P(-1, 1, -1)$ $\nabla \times \mathrm{A} = -4\mathrm{i} + 4\mathrm{k}$

$$\text{(ii) } \nabla \cdot \mathrm{A} = \left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right) \cdot (xy^2\mathrm{i} + x^2yz\mathrm{j} + y^3z\mathrm{k})$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(x^2yz) + \frac{\partial}{\partial z}(y^3z)$$

$$= y^2 + x^2z + y^3|_{p(-1,1,-1)} = 1 - 1 + 1 = 1$$

Note If $F(x, y, z) = c$ is the equation of a surface then ∇F is perpendicular to the surface. that is, normal vector to the surface at the point $p(x, y, z)$. A surface simply mean a solid which is represented by $x^2 + y^2 = r^2$ circle, $x^2 + y^2 + z^2 = r^2 sphere$, $y^2 = 4ax$ parabola .

Proof

Let $\mathrm{r} = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ be position vector on the surface F.

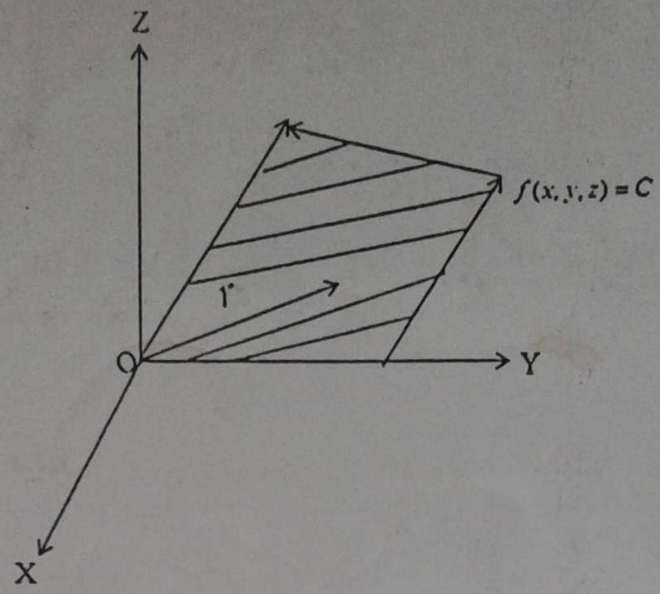

Figure 1.3:

$dr = dx\mathrm{i} + dy\mathrm{j} + dz\mathrm{k}$ is a tangent to the surface

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

$$\left(\frac{\partial F}{\partial x}\mathrm{i} + \frac{\partial F}{\partial y}\mathrm{j} + \frac{\partial F}{\partial z}\mathrm{k}\right) \cdot (dx\mathrm{i} + dy\mathrm{j} + dz\mathrm{k}) = 0$$

$$= \nabla F \cdot \mathrm{dr} = 0$$

(which implies that grad F is perpendicular to the tangent)
Hence ∇F is perpendicular to the surface $F(x, y, z) = c$ infact ∇F is a normal vector to the surface.

Example 1.7

Find a unit normal vector to the surface $5x^2 - 7xy + 5z^2 = 0$ at the point $P(2, -1, 3)$.

Solution

Let $F = 5x^2 - 7xy + 5z^2$

$$\nabla F = \left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right) \cdot (5x^2 - 7xy + 5z^2)$$

$$(10x - 7y)\mathbf{i} + (-7x)\mathbf{j} + 10z\mathbf{k}$$

$\therefore$ at $P(2, -1, 3)$

$$= 27\mathbf{i} - 14\mathbf{j} + 30\mathbf{k}$$

$\therefore$ the unit normal vector to the surface is

$$\frac{27\mathbf{i} - 14\mathbf{j} + 30\mathbf{k}}{\sqrt{27^2 + 14^2 + 30^2}} = \frac{27\mathbf{i} - 14\mathbf{j} + 30\mathbf{k}}{\sqrt{1825}} = \frac{27\mathbf{i} - 14\mathbf{j} + 30\mathbf{k}}{5\sqrt{73}}$$

1.3.4 Equation of Tangent Plane and Normal Line

Definition

A tangent plane to a surface F is a plane which touches the surface at only one point

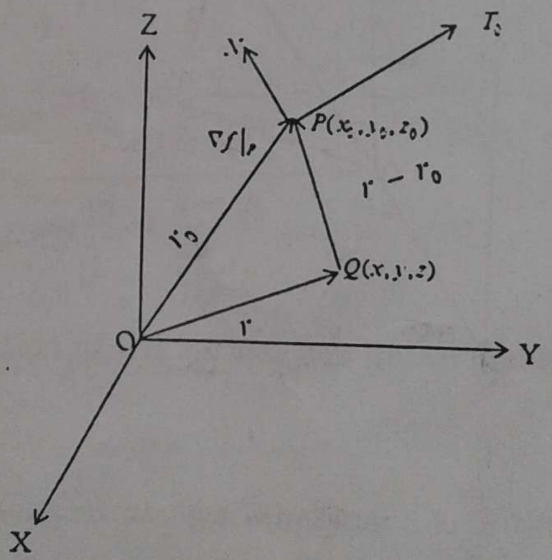

Figure 1.4:

Let $\mathbf{r}_o = x_o\mathbf{i} + y_o\mathbf{j} + z_o\mathbf{k}$ be position vector of a point $P(x_o, y_o, z_o)$ from the origin O.
Similarly, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point $Q(x, y, z)$ from O.
Let T_o be the tangent plane at $P(x_o, y_o, z_o)$ the vector $\mathbf{r} - \mathbf{r}_0 = (x - x_0)\mathbf{i} + (y -$

$y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ is perpendicular to the tangent T_0 and therefore $(\mathbf{r} - \mathbf{r}_0) \cdot N_0 = (\mathbf{r} - \mathbf{r}_0) \cdot \nabla F|_P = 0$ by expanding we have

$$[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] \cdot \left(\mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z}\right) = 0$$

$$(x - x_0)\frac{\partial F}{\partial x}|_P + (y - y_0)\frac{\partial F}{\partial y}|_P + (z - z_0)\frac{\partial F}{\partial z}|_P = 0$$

The normal line is given by

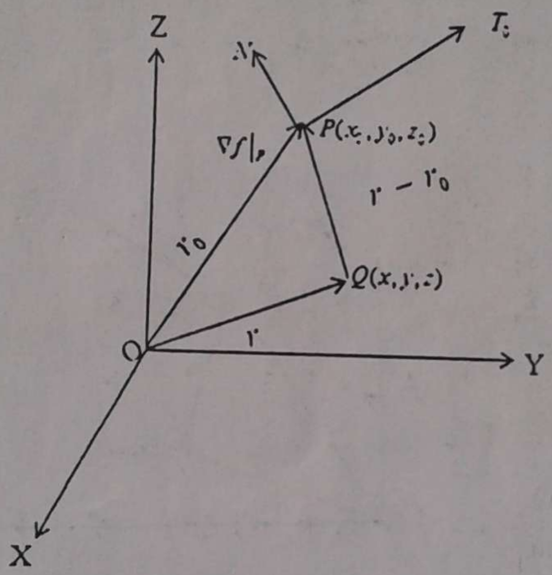

Figure 1.5:

$(\mathbf{r} - \mathbf{r}_0) \times N_0 = (\mathbf{r} - \mathbf{r}_0) \times \nabla F|_P = 0$ by expanding we have

$$[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] \times \left(\mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z}\right) = 0$$

$$(x - x_0)\frac{\partial F}{\partial y}\mathbf{k} - (x - x_0)\frac{\partial F}{\partial z}\mathbf{j} - (y - y_0)\frac{\partial F}{\partial x}\mathbf{k} + (y - y_0)\frac{\partial F}{\partial z}\mathbf{i}$$

$$+(z - z_0)\frac{\partial F}{\partial x}\mathbf{j} - (z - z_0)\frac{\partial F}{\partial y}\mathbf{i} = 0$$

$$\left[(y - y_0)\frac{\partial F}{\partial z} - (z - z_0)\frac{\partial F}{\partial y}\right]\mathbf{i} + \left[(z - z_0)\frac{\partial F}{\partial x} - (x - x_0)\frac{\partial F}{\partial z}\right]\mathbf{j}$$

$$+\left[(x-x_0)\frac{\partial F}{\partial y}-(y-y_0)\frac{\partial F}{\partial x}\right]\mathbf{k}=0$$

The above vector is zero if and only if

$$(y-y_0)\frac{\partial F}{\partial z}-(z-z_0)\frac{\partial F}{\partial y}=0\Rightarrow\frac{\frac{\partial F}{\partial y}}{y-y_0}=\frac{\frac{\partial F}{\partial z}}{z-z_0}\tag{1.1}$$

Similarly

$$\frac{\frac{\partial F}{\partial x}}{x-x_0}=\frac{\frac{\partial F}{\partial z}}{z-z_0}\tag{1.2}$$

and

$$\frac{\frac{\partial F}{\partial x}}{x-x_0}=\frac{\frac{\partial F}{\partial y}}{y-y_0}\tag{1.3}$$

the three equations gives

$$\frac{\frac{\partial F}{\partial x}|_P}{x-x_0}=\frac{\frac{\partial F}{\partial y}|_P}{y-y_0}=\frac{\frac{\partial F}{\partial z}|_P}{z-z_0}$$

or

$$\frac{x-x_0}{\frac{\partial F}{\partial x}|_P}=\frac{y-y_0}{\frac{\partial F}{\partial y}|_P}=\frac{z-z_0}{\frac{\partial F}{\partial z}|_P}$$

which is the require equation of the normal line to the plane.

Example 1.8

Find the equation of the tangent plane and normal line to the surface $2xy^2z+y^2x^2=5$ at $(-1,3,-1)$.

Solution

Let $F(x,y,z)=2xy^2z+y^2x^2-5$

$\frac{\partial f}{\partial x}=2y^2z+2xy^2$ at $(-1,3,-1)$, $\frac{\partial f}{\partial x}=2(3)^2(-1)+2(-1)(3)^2=-18-18=-36$

$\frac{\partial f}{\partial y}=4xyz+2x^2y$ at $(-1,3,-1)$, $\frac{\partial f}{\partial y}=4(-1)(3)(-1)+2(-1)^2(3)=12+6=18$

$$\frac{\partial f}{\partial z} = 2xy^2 \ at \ (-1,3,-1), \ \frac{\partial f}{\partial z} = 2(-1)(3)^2 = -18$$

Therefore the equation of the tangent plane is

$$(x+1)(-36) + (y-3)(18) + (z+1)(-18) = 0$$

$$\Rightarrow 2(x+1) - (y-3) + (z+1) = 0$$

2x − y + z = −6

The equation of the normal line is

$$\frac{x+1}{-36} = \frac{y-3}{18} = \frac{z+1}{-18}$$

$$\Rightarrow \frac{x+1}{2} = \frac{y-3}{-1} = \frac{z+1}{1}$$

Example 1.9

Show that the surfaces $x^2 = 2yz - y^3 + 4$ is perpendicular to the family of surfaces $x^2 + (4a-2)y^2 = az^2 - 1$ at the point $(1,-1,2)$, where a is constant.

Solution

Note: two surfaces $F$and G are said to be perpendicular if and only if their Normal lines are perpendicular. That is, $\nabla F|_p \cdot \nabla G|_P = 0$.

Let $F = x^2 - 2yz + y^3 - 4 = 0$ and $G = x^2 + (4a-2)y^2 - az^2 + 1 = 0$

$$\nabla F = \left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right)(x^2 - 2yz + y^3 - 4)$$

$$= 2x\mathrm{i} + (3y^2 - 2z)\mathrm{j} - 2y\mathrm{k}$$

$$\nabla F|_{(1,-1,2)} = 2\mathrm{i} - \mathrm{j} + 2\mathrm{k}$$

$$\nabla G = \left(\mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}\right)(x^2 + (4a-2)y^2 - az^2 + 1)$$

$$= 2x\mathrm{i} + 2y(4a-2)\mathrm{j} - 2az\mathrm{k}$$

$$\nabla G|_{(1,-1,2)} = 2\mathrm{i} - 2(4a-2)\mathrm{j} - 4a\mathrm{k}$$

$$\nabla F|_p \cdot \nabla G|_P = (2\mathrm{i} - \mathrm{j} + 2\mathrm{k}) \cdot (2\mathrm{i} - 2(4a-2)\mathrm{j} - 4a\mathrm{k})$$

$$= 4 + 2(4a-2) - 8a = 4 + 8a - 4 - 8a = 0$$

$\therefore$ The surfaces are orthogonal or perpendicular.

Exercise 1.2

1. Find the equation of the tangent plane and Normal line to the following surfaces at the indicated points
   (a) $x^3 + y^3 + z^3 = 36$ at $(1, 2, 3)$
   (b) $x^2 - y^2 + z^2 = 25$ at $(3, 0, 4)$
   (c) $xyz^2 - y^2 + xz = -1$ at $(1, -1, 1)$
   (d) $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$

2. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$.

3. Prove that
$$\nabla \cdot (U\nabla V - V\nabla U) = U\nabla^2 V - V\nabla^2 U$$

4. Prove that
$$\nabla \times (\nabla \times \mathrm{A}) = -\nabla^2 \mathrm{A} + \nabla(\nabla \cdot \mathrm{A})$$

5. If $\mathrm{V} = \mathrm{w} \times \mathrm{r}$ Prove that $\mathrm{w} = \frac{1}{2} Curl \mathrm{V}$ where w is a constant vector.

6. If the vector $\mathrm{V} = (x + 2y + az)\mathrm{i} + (bx - 3y - z)\mathrm{j} + (4x + cy + 2z)\mathrm{k}$ is irrotational, find the values of a, b and c.

1.4 Multiple Integrals

1.4.1 Double Integral

Suppose the function $f(x, y)$ is defined in some region R of the xy plane that has finite area(see Fig. 1.6)
$a_0 \leq x \leq a_1$ and $b_0 \leq y \leq b_1$
Divide R into n subregions of $\triangle A_1, \triangle A_2, \cdots, \triangle A_n$. In each subregion $\triangle A_i (i = 1, 2, \cdots, n)$ we chose a point (x_i, y_i) and form the sum.

$$f(x_1, y_1)\triangle A_1 + f(x_2, y_2)\triangle A_2 + \cdots, + f(x_n, y_n)\triangle A_n = \sum_{i=1}^{n} f(x_i, y_i)\triangle A_i.$$

Taking limit as $\triangle A_i \to 0$ gives

$$= \int_R f(x, y) dA = \iint_R f(x, y) dx dy \tag{1.4}$$

Integral over R

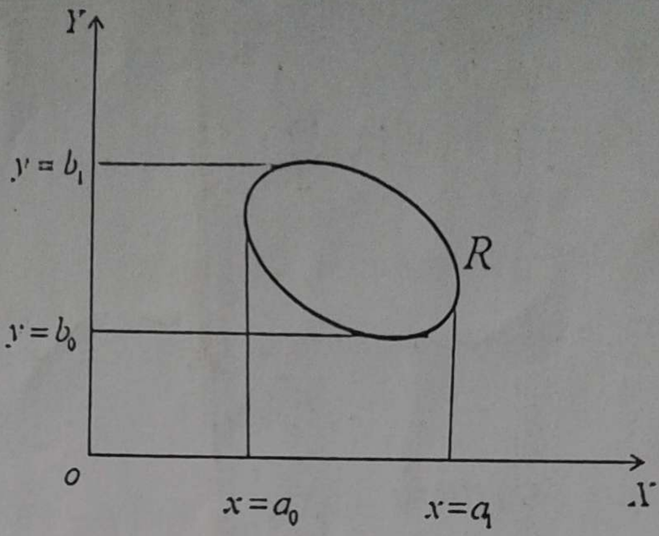

Figure 1.6:

eqn(1.4) is called the double integral of $f(x, y)$ over R. The double integral over R denotes Area bounded by the region.

1.4.2 Iterated Integral

Iterated integrals are used to evaluates double integrals. If R is such that any line parallel to the $y-axis$ meets the boundary of R in at most two points and also if R is bounded by the curves $y = f_1(x) or y = f_2(x)$ respectively where $f_1(x) or f_2(x)$ are single valued in $a \leq x \leq b$ then

$$\iint_R f(x, y)dxdy = \int_{x=a}^{x=b}\left[\int_{y=f_1(x)}^{y=f_2(x)} f(x, y)dy\right]dx.$$

Example 1.10

Evaluate the double integral

$$\int_1^5\int_0^4 x^2y\,dxdy$$

Solution

$$\int_1^5\int_0^4 x^2ydxdy = \int_1^5\left[\int_0^4 x^2ydx\right]dy.$$

$$= \int_1^5 \left[\frac{x^3 y}{3}\right]_0^4 dy = \int_1^5 \left[\frac{64y}{3}\right] dy.$$

$$= \frac{64}{3}\left[\frac{y^2}{2}\right]_1^5 = \frac{64}{3}\left[\frac{5^2}{2} - \frac{1^2}{2}\right]$$

$$= \frac{64}{3}\left[\frac{25}{2} - \frac{1}{2}\right] = \frac{64}{3}\left[\frac{24}{2}\right] = 256$$

Example 1.11

Evaluate

$$\iint_R (x^2 y + y^2) dx dy.$$

Where R is bounded by $y = x^2$ and $x = y^2$

Solution

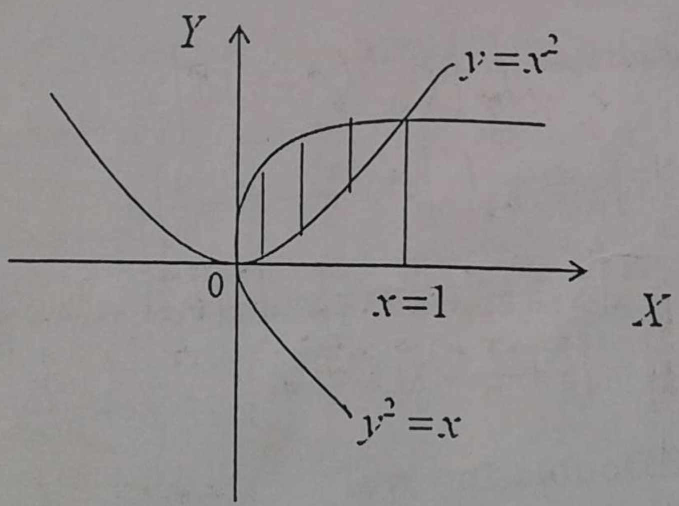

Figure 1.7:

$y^2 = x^4 = x \Rightarrow x - x^4 = 0 \Rightarrow x(1 - x^3) = 0 \Rightarrow x = 1 or x = 0$ then

$$\iint_R (x^2 y + y^2) dx dy = \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x^2 y + y^2) dy dx.$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{y^3}{3}\right]_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \left[\frac{x^3}{2} + \frac{x\sqrt{x}}{3} - \frac{x^6}{2} - \frac{x^6}{3}\right] dx$$
$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^{3/2}}{3} - \frac{5x^6}{6}\right] dx = \left[\frac{x^4}{8} + \frac{2x^{5/2}}{15} - \frac{5x^7}{42}\right]_0^1$$
$$= \frac{1}{8} + \frac{2}{15} - \frac{5}{42} = \frac{105 + 112 - 100}{840} = \frac{117}{840} = \frac{39}{280}$$

Example 1.12

Evaluate

$$\iint_R (x^2 + y^2)dxdy$$

Where R in xy plane bounded by $y = x^2, x = 2, y = 1$.

Solution

$$\iint_R (x^2 + y^2)dxdy = \int_1^2 \left[\int_{y=1}^{y=x^2} (x^2 + y^2)dy\right] dx.$$
$$= \int_1^2 \left[x^2y + \frac{y^3}{3}\right]_1^{x^2} dx = \int_1^2 \left[x^4 + \frac{x^6}{3} - x^2 - \frac{1}{3}\right] dx$$
$$= \left[\frac{x^5}{5} + \frac{x^7}{21} - \frac{x^3}{3} - \frac{x}{3}\right]_1^2 = \left[\frac{2^5}{5} + \frac{2^7}{21} - \frac{2^3}{3} - \frac{2}{3}\right] - \left[\frac{1^5}{5} + \frac{1^7}{21} - \frac{1^3}{3} - \frac{1}{3}\right]$$
$$= \left[\frac{32}{5} + \frac{128}{21} - \frac{8}{3} - \frac{2}{3}\right] - \left[\frac{1}{5} + \frac{1}{21} - \frac{2}{3}\right] = \frac{31}{5} + \frac{127}{21} - \frac{8}{3} = \frac{1006}{105}$$

1.4.3 Properties of Double Integral

If $\int_R \int f(x,y)dxdy$ over R exist, and k is a constant. then

1.
$$\iint_R kf(x,y)dxdy = k\iint_R f(x,y)dxdy.$$

2.
$$\iint_R (f(x,y) + g(x,y))dxdy = \iint_R f(x,y)dxdy + \iint_R g(x,y)dxdy$$

3. if R is subdivided into two region R_1 and R_2
$$\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$$

Exercise 1.3

(a) Evaluate.

$$\int_0^2 \int_0^{\pi/4} x^2 \cos y dy dx.$$

(b) Evaluate

$$\iint_R (4 - x^3 y) dA$$

where R is bounded by $x = \sqrt{4-y}, x = 1, y = 1$

(c) Evaluate the integral

$$\int_{y=0}^{2} \int_{x=0}^{y^2/2} \frac{y}{\sqrt{x^2+y^2+1}} dy dx$$

1.4.4 Applications of Double Integrals

(A) If the function $f(x, y) = 1$ then the integral

$$\iint_R dx dy,$$

denote the area bounded by the region R.

Example 1.13

Evaluate $\iint_R$ where R is bounded by $1 \le x \le 5, 3 \le y \le 7$

Solution

See figure 1.8

$$\int_{x=1}^{5} \int_{y=3}^{7} dy dx = \int_{x=1}^{5} [y]_3^7 \, dx = \int_{x=1}^{5} (7-3) dx = \int_{x=1}^{5} 4 dx$$

$$= 4x|_1^5 = 4(5-1) = 16 squareunits$$

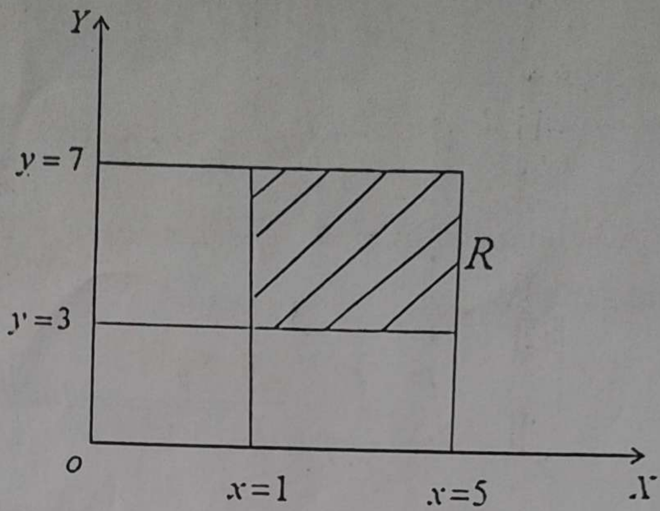

Figure 1.8:

Example 1.14

Find the area bounded by the curve $x = y^2$ and $y = x^2$

Solution

See figure 1.9
$y^2 = x^4 = x \Rightarrow x - x^4 = 0 \Rightarrow x(1 - x^3) = 0 \Rightarrow x = 1 or x = 0$ then

$$A = \int_{x=0}^{1} \int_{y=x^2}^{y=\sqrt{x}} dx dy = \int_0^1 y|_{y=x^2}^{\sqrt{x}} dx$$

(B) Let $f(x, y) = \rho(x, y)$ where $\rho(x, y)$ is a density function per unit area of a body bounded by R. Then the mass of the body is given by

$$m = \iint_R \rho dx dy$$

and the centroid of the body is given by $(\bar{x}, \bar{y})$ and

$$\bar{x} = \frac{1}{m} \iint_R x\rho dx dy$$

$$\bar{y} = \frac{1}{m} \iint_R y\rho dx dy$$

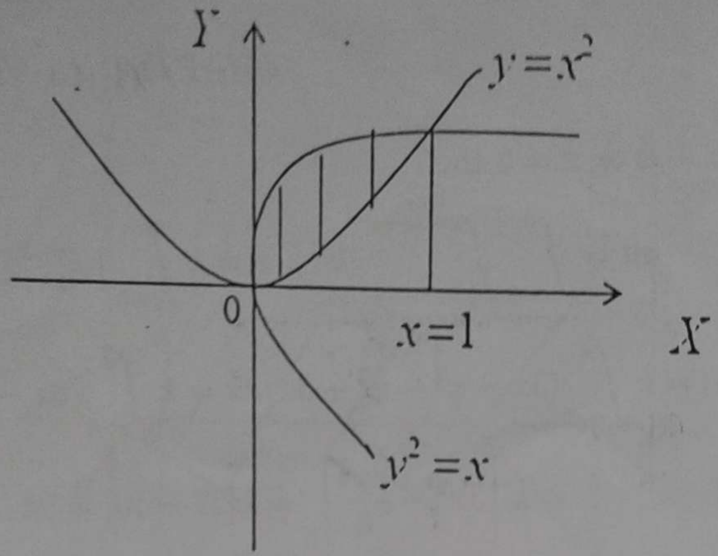

Figure 1.9:

Example 1.15

Find the mass of the body bounded by the curves $y = 3x - x^2$ and $y = x^2 - 3x$ if the density is given by $\rho(x, y) = k$.

Solution

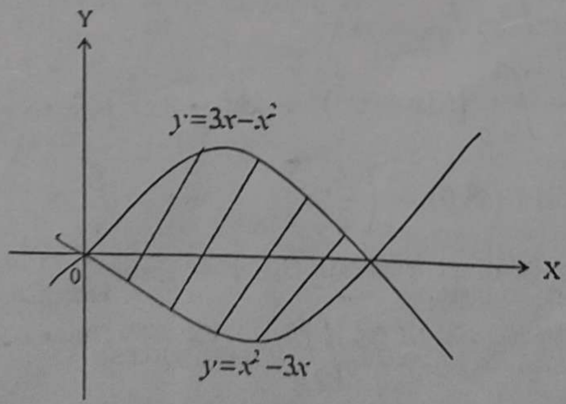

Figure 1.10:

$3x - x^2 = x^2 - 3x \Rightarrow 2x^2 - 6x = 0 \Rightarrow x^2 - 3x = 0 \Rightarrow x(x - 3) =$

$0 \Rightarrow x = 0$ *or* $x = 3$ then

$$m = \int_{x=0}^{3}\int_{y=x^2-3x}^{y=3x-x^2} kdydx = k\int_{x=0}^{3} [y]_{y=x^2-3x}^{y=3x-x^2} dx$$

$$= k\int_{x=0}^{3} (3x - x^2 - x^2 + 3x)dx = k\int_{x=0}^{3} (6x - 2x^2)dx$$

$$= k\left[3x^2 - \frac{2x^3}{3}\right]_0^3 = k[27 - 18] = 9k$$

$$\bar{x} = \frac{1}{m}\int_{x=0}^{3}\int_{y=x^2-3x}^{y=3x-x^2} xkdydx = \frac{k}{m}\int_{x=0}^{3} xy|_{y=x^2-3x}^{y=3x-x^2}dx$$

$$= \frac{k}{m}\int_{x=0}^{3}(x(3x - x^2) - x(x^2 - 3x))dx = \frac{k}{m}\int_{x=0}^{3}(6x^2 - 2x^3)dx$$

$$= \frac{k}{m}\left[2x^3 - \frac{x^4}{2}\right]_{x=0}^{3} = \frac{k}{m}\left[54 - \frac{81}{2}\right] = \frac{k}{m}\left[\frac{108 - 81}{2}\right] = \frac{27k}{2m}$$

but $m = 9k$

$$\bar{x} = \frac{27k}{18k} = \frac{3}{2}$$

$$\bar{y} = \frac{1}{m}\int_{x=0}^{3}\int_{y=x^2-3x}^{y=3x-x^2} ykdydx = \frac{k}{m}\int_{x=0}^{3}\left[\frac{y^2}{2}\right]_{y=x^2-3x}^{y=3x-x^2} dx$$

$$= \frac{k}{2m}\int_{x=0}^{3}\left[(3x - x^2)^2 - (x^2 - 3x)^2\right] dx = \frac{k}{2m}(0) = 0$$

$\therefore$ The centroid $(\bar{x}, \bar{y}) = \left(\frac{3}{2}, o\right)$

(C) The moment of inertia about the $x - axis$

$$I_x = \iint_R y^2\rho(x, y)dxdy$$

about $y - axis$

$$I_y = \iint_R x^2\rho(x, y)dxdy$$

Example 1.16

Find the moment of inertia of the body bounded by the curves $y = 2x - x^2$ and $y = x^2 - 2x$

Solution

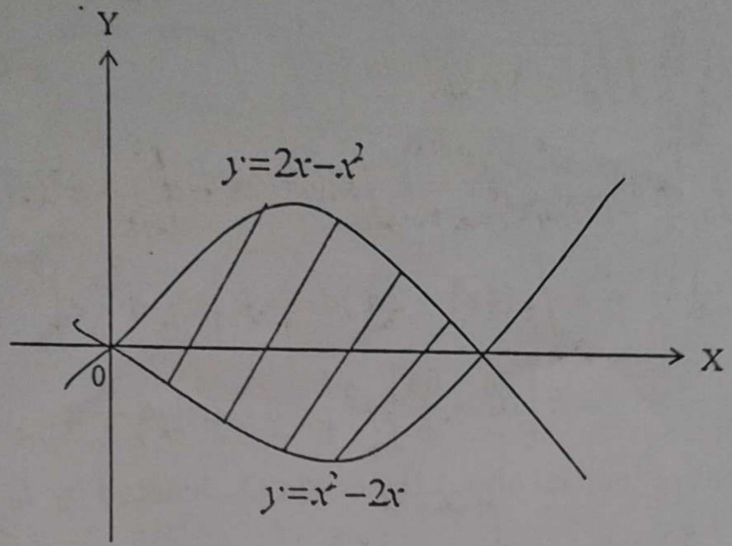

Figure 1.11:

$2x - x^2 = x^2 - 2x \Rightarrow 2x^2 - 4x = 0 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow$
$x = 0$ *or* $x = 2$ Let $\rho(x, y) = k$ then

$$I_x = \iint_R y^2 \rho(x,y) dxdy = \int_0^2 \int_{y=x^2-2x}^{y=2x-x^2} y^2 k dydx$$

$$= k \int_0^2 \left[\frac{y^3}{3}\right]_{y=x^2-2x}^{y=2x-x^2} dx = \frac{k}{3} \int_0^2 \left[(2x - x^2)^3 - (x^2 - 2x)^3\right] dx$$

$$= \frac{k}{3} \int_0^2 [8x^3 - 12x^4 + 6x^5 - x^6 - (x^6 - 6x^5 + 12x^4 - 8x^3)]dx$$

$$= \frac{k}{3} \int_0^2 (16x^3 - 24x^4 + 12x^5 - 2x^6)dx = \frac{2k}{3}\left[2x^4 - \frac{12x^5}{5} + x^6 - \frac{x^7}{7}\right]_0^2$$

$$= \frac{2k}{3}\left[32 - \frac{12 \times 32}{5} + 64 - \frac{128}{7}\right]$$

$$= \frac{2k}{3}\left[\frac{32 \times 35 - 12 \times 7 + 64 \times 35 - 128 \times 5}{35}\right]$$

$$= \frac{2k}{105}[1120 - 2668 + 2240 - 640] = \frac{64}{105}k$$

$$I_y = \iint_R x^2\rho(x,y)dxdy = \int_0^2\int_{y=x^2}^{y=2x-x^2-2x} x^2kdydx$$

$$= k\int_0^2 [x^2y]_{y=x^2-2x}^{y=2x-x^2}dx = k\int_0^2 [(x^2(2x-x^2)-x^2(x^2-2x)]dx$$

$$= k\int_0^2 (4x^3-2x^4)dx = k\left[x^4-\frac{2x^5}{5}\right]_0^2$$

$$= k\left[16-\frac{64}{5}\right] = \frac{k}{5}[80-64] = \frac{16}{5}k$$

The moment of inertial about the origin O, denoted by $I_0 = I_x + I_y$ that is,

$$I_0 = \frac{64}{105}k + \frac{16}{5}k = \frac{k}{105}(64+336) = \frac{400}{105}k = \frac{80}{21}k$$

Exercise 1.4

1. Evaluate the integral

$$\int_{y=0}^2\int_{x=1}^{\sqrt{4-y}}(x+y)dxdy$$

2. (a) find the area of the xy plane bounded by $y^2 = 2x$ and $y = x$
   (b) moment of inertia of R assuming constant density k
   (c) the centroid.

1.4.5 Triple Integral

Suppose a function $f(x, y, z)$ is defined in a closed region of a volume V. (See Fig.1.12)
$\triangle V_i = f(x_i, y_i, z_i)$ Let $f(x_i, y_i, z_i)$ be the value of the function at the point (x_i, y_i, z_i) in the region, then we form the sum

$$\sum_{i=1}^n f(x_i, y_i, z_i)\triangle V_i$$

That is,

$$f(x_1,y_1,z_1)\triangle V_1 + f(x_2,y_2,z_2)\triangle V_2 + \cdots + f(x_n,y_n,z_n)\triangle V_n = \sum_{i=1}^n f(x_i,y_i,z_i)\triangle V_i$$

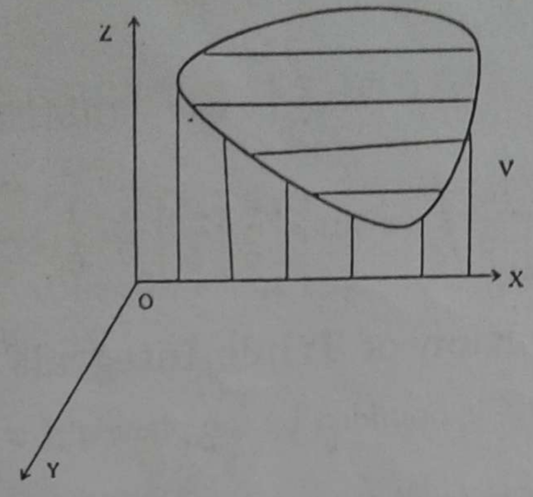

Figure 1.12:

$$\lim_{\Delta V_i \to o} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \int_V f(x_i, y_i, z_i)dV = \iiint_V f(x_i, y_i, z_i)dxdydz$$

The above is called the triple integral of $f(x, y, z)$ over V.

1.4.6 Evaluation of triple integrals

If the volume V is bounded by the plane $a \leq x \leq b$, $h_1(x) \leq y \leq h_2(x)$, and $g_1(x.y) \leq z \leq g_2(x, y)$ the the triple integral becomes

$$\int_{x=a}^{b} \left[\int_{y=h_1(x)}^{h_2(x)} \left[\int_{z=g_1(x,y)}^{g_2(x,y)} f(x, y, z)dz \right] dy \right] dx$$

Example 1.17

Evaluate

$$\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} xydzdydx$$

Solution

$$\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{x+y} xydzdydx = \int_{x=0}^{1} \int_{y=0}^{x} xy\,[z]_{z=0}^{x+y}\,dydx$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (x^2y + xy^2)dydx = \int_{x=0}^{1} \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{y=0}^{x} dx$$

$$= \frac{5}{6}\int_0^1 (x^4)dx = \frac{5}{6}\left[\frac{x^5}{5}\right]_0^1 = \frac{1}{6}(1-0) = \frac{1}{6}$$

1.4.7 Application of Triple Integrals

(A) If the region V is bounded by the plane $a \le x \le b$, $h_1(x) \le y \le h_2(x)$, and
$g_1(x,y) \le z \le g_2(x,y)$ then the triple integral denotes the volume of the region.

Example 1.18

Find the volume (of the region) bounded by the plane $z = x^2 + y$, $y = 2x$, $z = 0$ $x = 0$, $y = 3$ and $x = 1$.

Solution

We have $o \le x \le 1$, $0 \le z \le x^2 + y$, $2x \le y \le 3$ then

$$V = \int_{x=0}^1 \int_{y=2x}^3 \int_{z=0}^{x^2+y} dzdydx = \int_{x=0}^1 \int_{y=2x}^3 [z]_0^{x^2+y} dydx$$

$$= \int_{x=0}^1 \int_{y=2x}^3 (x^2+y)dydx = \int_{x=0}^1 \left[x^2y + \frac{y^2}{2}\right]_{y=2x}^3 dx$$

$$= \int_0^1 \left[3x^2 + \frac{9}{2} - 2x^3 - \frac{4x^2}{2}\right] dx = \left[x^3 + \frac{9x}{2} - \frac{x^4}{2} - \frac{2x^3}{3}\right]_0^1$$

$$= 1 + \frac{9}{2} - \frac{1}{2} - \frac{2}{3} = \frac{6+27-3-4}{6} = \frac{26}{6} = \frac{13}{3} \textit{cubic unit}$$

(B) If $f(x,y,z) = \rho(x,y,z)$ is the density of the body bounded by the region V. Then the triple integrals denoted by the mass m of the body.

$$m = \iiint_V \rho(x,y,z)dxdydz$$

The centre of mass(centroid) $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{m}\iiint_V x\rho(x,y,z)dxdydz$$

$$\bar{y} = \frac{1}{m}\iiint_V y\rho(x, y, z)dxdydz$$

$$\bar{z} = \frac{1}{m}\iiint_V z\rho(x, y, z)dxdydz$$

(C) The moment of inertia about the x-axis, y-axis and z-axis are given respectively by

$$I_x = \iiint_V (y^2 + z^2)\rho(x, y, z)dxdydz$$

$$I_y = \iiint_V (x^2 + z^2)\rho(x, y, z)dxdydz$$

$$I_z = \iiint_V (x^2 + y^2)\rho(x, y, z)dxdydz$$

Example 1.19

Find the mass of an object of constant density ρ if it is bounded by the surface $z = 1 - 2y^2$, $x = 0$, $z = 0$, $x = 2$, $y = 0$, $y = 1$. Also find the centroid and moment of inertia of the object.

Solution

$$m = \iiint \rho dxdydz = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1-2y^2} \rho dzdydx$$

$$= \rho\int_{x=0}^{2}\int_{y=0}^{1} [z]_0^{1-2y^2} dydx = \rho\int_{x=0}^{2}\int_{y=0}^{1} (1 - 2y^2)dydx$$

$$= \rho\int_{x=0}^{2}\left[y - \frac{2y^3}{3}\right]_{y=0}^{1} dx = \rho\int_{x=0}^{2}\left[1 - \frac{2}{3}\right] dx = \frac{\rho}{3}\int_{x=0}^{2} dx$$

Mass $$= \frac{\rho}{3}[x]_0^2 = \frac{2\rho}{3}$$

To calculate the centroid $(\bar{x}, \bar{y}, \bar{z})$ we have

$$\bar{x} = \frac{\rho}{m}\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1-2y^2} xdzdydx = \frac{\rho}{m}\int_{x=0}^{2}\int_{y=0}^{1} x(1 - 2y^2)dydx$$

$$= \frac{\rho}{m}\int_{x=0}^{2}\left[x\left(y - \frac{2y^3}{3}\right)\right]_{y=0}^{1} dx = \frac{\rho}{m}\int_{x=0}^{2} x\left(1 - \frac{2}{3}\right) dx = \frac{\rho}{3m}\int_{x=0}^{2} xdx$$

$$= \frac{\rho}{3m}\left[\frac{x^2}{2}\right]_0^2 = \frac{\rho}{3m}[2] = \frac{2\rho}{3m}$$

But $m = \dfrac{2\rho}{3}$ therefore, $\bar{x} = \dfrac{2\rho}{3\left(\dfrac{2\rho}{3}\right)} = 1$

$$\bar{y} = \frac{\rho}{m}\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1-2y^2} y\,dz\,dy\,dx = \frac{\rho}{m}\int_{x=0}^{2}\int_{y=0}^{1} y(1-2y^2)\,dy\,dx$$

$$= \frac{\rho}{m}\int_{x=0}^{2}\left[\frac{y^2}{2} - \frac{y^4}{2}\right]_0^1 dx = 0$$

therefore $\bar{y} = 0$

$$\bar{z} = \frac{\rho}{m}\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1-2y^2} z\,dz\,dy\,dx = \frac{\rho}{m}\int_{x=0}^{2}\int_{y=0}^{1}\left[\frac{z^2}{2}\right]_0^{(1-2y^2} dy\,dx$$

$$= \frac{\rho}{2m}\int_{x=0}^{2}\int_{y=0}^{1}(1-2y^2)^2\,dy\,dx = \frac{\rho}{2m}\int_{x=0}^{2}\int_{y=0}^{1}(1-4y^2+4y^4)\,dy\,dx$$

$$= \frac{\rho}{2m}\int_{x=0}^{2}\left[y - \frac{4y^3}{3} + \frac{4y^5}{5}\right]_0^1 dx = \frac{\rho}{2m}\int_{x=0}^{2}\left[1 - \frac{4}{3} + \frac{4}{5}\right] dx$$

$$= \frac{\rho}{2m}\left[\frac{15-20+12}{15}\right][x]_0^2 = \frac{\rho}{2m}\left[\frac{7}{15}\right]\times 2 = \frac{\rho}{m}\left[\frac{7}{15}\right]$$

But $m = \dfrac{2\rho}{3}$ therefore, $\bar{z} = \dfrac{\rho}{\left(\dfrac{2\rho}{3}\right)}\left[\dfrac{7}{15}\right] = \dfrac{3}{2}\left[\dfrac{7}{15}\right] = \dfrac{7}{10}$

$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(1, 0, \dfrac{7}{10}\right).$

To find moment of inertia about the x-axis we have

$$I_x = \rho\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1-2y^2}(y^2+z^2)\,dz\,dy\,dx = \rho\int_{x=0}^{2}\int_{y=0}^{1}\left[y^2z + \frac{z^3}{3}\right]_{z=0}^{1-2y^2} dy\,dx$$

$$= \rho\int_{x=0}^{2}\int_{y=0}^{1}\left[y^2(1-2y^2) + \frac{(1-2y^2)^3}{3}\right] dy\,dx$$

$$= \frac{\rho}{3}\int_{x=0}^{2}\int_{y=0}^{1}\left[3y^2 - 6y^4 + 1 - 6y^2 + 12y^4 - 8y^6\right] dy\,dx$$

$$= \frac{\rho}{3}\int_{x=0}^{2}\int_{y=0}^{1}\left[1 - 3y^2 + 6y^4 - 8y^6\right] dydx = \frac{\rho}{3}\int_{x=0}^{2}\left[y - y^3 + \frac{6y^5}{5} - \frac{8y^7}{7}\right]_{y=0}^{1} dx$$

$$= \frac{\rho}{3}\int_{x=0}^{2}\left[1 - 1 + \frac{6}{5} - \frac{8}{7}\right] dx = \frac{\rho}{3}\int_{x=0}^{2}\left[\frac{42-40}{35}\right] dx = \frac{2\rho}{105}[x]_0^2 = \frac{4\rho}{105}$$

Compute I_y, I_z and I_0 as an exercise.

1.5 Transformation of Multiple Integrals

Sometimes it may be expedient to transform the given integrals to a new one involving different variables in order to reduce the complexity of the given integrals. In this case the integrals are expressed in terms of new coordinates system for instance, the polar coordinates, cylindrical coordinates and spherical coordinates system.

1.5.1 Transformation of double integrals

The double integrals

$$\iint_R f(x,y)dxdy$$

can be transformed into a new integrals using the transformation variable (u, v) that is, $x = x(u,v)$, $y = y(u,v)$ as

$$\iint_R f(x,y)dxdy = \iint_R f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

The determinant

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Example 1.20

If the above double integral is to be transformed using the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le 2\pi$.

Solution

The Jacobian of this transformation is

$$J = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Therefore

$$\iint_R f(x,y)dxdy = \int_{r_1}^{r_2}\int_{\theta=0}^{2\pi} f(r,\theta)rdrd\theta.$$

Example 1.21

Evaluate the integrals

$$\iint_R \sqrt{x^2+y^2}dxdy$$

where R is bounded by $x^2+y^2=9$ and $x^2+y^2=25$.

Solution

Transforming the integral into polar form gives $r = \sqrt{x^2+y^2}$ since $x = r\cos\theta,\ y = r\sin\theta,\ 0 \le \theta \le 2\pi.$ and

$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2} = r$$

$$x^2+y^2 = r^2 = 9 \Rightarrow r = 3$$

$$x^2+y^2 = r^2 = 25 \Rightarrow r = 5$$

$$\int_R\int \sqrt{x^2+y^2}dxdy = \int_{\theta=0}^{2\pi}\int_{r=3}^{5} r\cdot rdrd\theta = \int_{\theta=0}^{2\pi}\left[\frac{r^3}{3}\right]_{r=3}^{5}$$

$$= \frac{1}{3}[125-27]\int_{\theta=0}^{2\pi} d\theta = \frac{98}{3}\times 2\pi = \frac{198}{3}\pi$$

Example 1.22

Evaluate

$$\int_{x=0}^{\infty} e^{-x^2}dx$$

Solution

Let

$$I = \int_{x=0}^{\infty} e^{-x^2} dx$$

$$I^2 = \int_{x=0}^{\infty} e^{-x^2} dx \int_{y=0}^{\infty} e^{-y^2} dy = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} dx dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $|J| = r$, $0 \le \theta \le \frac{\pi}{2}$. since the graph of e^{-x^2} lies in the first quadrant.

$$I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r dr d\theta = \int_{r=0}^{\infty} re^{-r^2} [\theta]_0^{\pi/2} dr = \frac{\pi}{2} \int_{r=0}^{\infty} re^{-r^2} dr$$

$$I^2 = -\frac{\pi}{4} \left[e^{-r^2}\right]_0^{\infty} = -\frac{\pi}{4} \left[e^{-\infty} - e^0\right] = \frac{\pi}{4}$$

Hence

$$I = \frac{\sqrt{\pi}}{2}$$

Therefore

$$\int_{x=0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Exercise 1.5

1. Evaluate the integral

$$\iint (x+y)^2 \cos^2(x-y) dx dy$$

by using the transformation $x = \frac{v-u}{2}$, $y = \frac{u+v}{2}$

2. Find the volume of the region bounded by the plane $z = x^2 + y^2$, $x = 1$, $z = 0$, $y = x$ and $x = 2y$.

3. Evaluate the integral

$$\int_0^a dx \int_0^{a-x} \frac{dy}{1+(x+y)^2}$$

by using the transformation $u = x + y, v = -x + y$.

Solution

Let

$$I = \int_{x=0}^{\infty} e^{-x^2} dx$$

$$I^2 = \int_{x=0}^{\infty} e^{-x^2} dx \int_{y=0}^{\infty} e^{-y^2} dy = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} dx dy$$

Let $x = r\cos\theta,\ y = r\sin\theta,\ |J| = r,\ 0 \le \theta \le \frac{\pi}{2}$. since the graph of e^{-x^2} lies in the first quadrant.

$$I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r dr d\theta = \int_{r=0}^{\infty} re^{-r^2} [\theta]_0^{\pi/2} dr = \frac{\pi}{2} \int_{r=0}^{\infty} re^{-r^2} dr$$

$$I^2 = -\frac{\pi}{4} \left[e^{-r^2}\right]_0^{\infty} = -\frac{\pi}{4} \left[e^{-\infty} - e^0\right] = \frac{\pi}{4}$$

Hence

$$I = \frac{\sqrt{\pi}}{2}$$

Therefore

$$\int_{x=0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Exercise 1.5

1. Evaluate the integral

$$\iint (x+y)^2 \cos^2(x-y) dx dy$$

by using the transformation $x = \frac{v-u}{2},\ y = \frac{u+v}{2}$

2. Find the volume of the region bounded by the plane $z = x^2 + y^2,\ x = 1,\ z = 0,\ y = x$ and $x = 2y$.

3. Evaluate the integral

$$\int_0^a dx \int_0^{a-x} \frac{dy}{1+(x+y)^2}$$

by using the transformation $u = x + y, v = -x + y$.

1.5.2 Transformation of Triple Integrals

If $x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w)$ then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \iiint_V f(x(u, v, w), y(u, v, w), z(u, v, w))|J|\,du\,dv\,dw$$

where

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Curvilinear coordinates

1. Cylindrical coordinates system

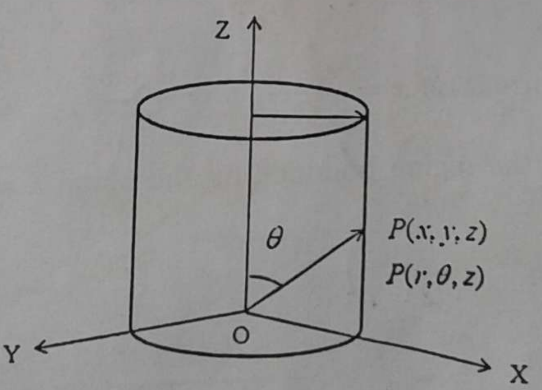

Figure 1.13:

That is, $x = r\cos\theta,\ y = r\sin\theta, z = z$

The Jacobian of the transformation is given by

$$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

In cylindrical polar coordinates of the triple integrals is

$$\iiint_V f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

2. Spherical coordinates system

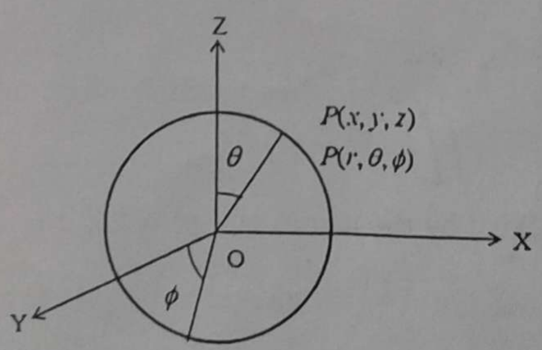

Figure 1.14:

That is, $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$, $0 \le r \le \infty$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$

The Jacobian of this transformation is given by

$$|J| = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$= \cos\phi(-r^2\sin^2\theta\sin\phi\cos\phi - r^2\cos^2\theta\sin\phi\cos\phi)$$
$$-r\sin\phi(r\cos^2\theta\sin^2\phi + r\sin^2\theta\sin^2\phi)$$
$$= -r^2\sin\phi\cos^2\phi(\sin^2\theta+\cos^2\theta) - r^2\sin^3\phi(\cos^2\theta+\sin^2\theta)$$
$$= -r^2\sin\phi\cos^2\phi - r^2\sin^3\phi = -r^2\sin\phi(\cos^2\phi+\sin^2\phi) = -r^2\sin\phi$$

Therefore the triple integral becomes

$$\iiint_V f(x,y,z)dxdydz = \iiint_V f(r,\theta,\phi)r^2\sin\phi drd\theta d\phi$$

Example 1.23

Evaluate the integral

$$\iiint_V (x^2+y^2)dV$$

where V is the region bounded by the planes $x^2+y^2=25$, $z=\sqrt{x^2+y^2}$, $x=0$, $y=0$, $z=0$.

Solution

Using the cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$ we have

$$\int_{x=0}^{5}\int_{y=0}^{\sqrt{25-x^2}}\int_{z=0}^{\sqrt{x^2+y^2}}(x^2+y^2)dxdydz = \int_{\theta=0}^{2\pi}\int_{r=0}^{5}\int_{z=0}^{r} r^2\cdot rdzdrd\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{5} r^3[z]_0^r drd\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{5} r^4 drd\theta = \int_{\theta=0}^{2\pi}\left[\frac{r^5}{5}\right]_0^5 d\theta$$

$$= 625\int_{\theta=0}^{2\pi} d\theta = 625(2\pi) = 1250\pi$$

Example 1.24

Evaluate

$$\iiint_V \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$$

where V is the region bounded by the spheres $x^2+y^2+z^2=9,\ x^2+y^2+z^2=25$ and $x\geq 0,\ y\geq 0,\ z\geq 0$

Solution

Let use spherical coordinates $x=r\cos\theta\sin\phi,\ y=r\sin\theta\sin\phi, z=r\cos\phi,\ 0\leq r\leq\infty,\ 0\leq\theta\leq 2\pi$, and $0\leq\phi\leq\pi\quad r=\sqrt{x^2+y^2+z^2}$

$$x^2+y^2+z^2=r^2=9\Rightarrow r=3$$

$$x^2+y^2=r^2+z^2=25\Rightarrow r=5$$

$$\iiint_V \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=3}^{5}\frac{1}{r^3}\cdot r^2\sin\phi drd\theta d\phi$$

$$=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}[\ln r]_3^5\sin\phi d\phi d\theta=\ln\left(\frac{5}{3}\right)\int_{\theta=0}^{2\pi}[-\cos\phi]_0^{\pi}d\theta$$

$$=2\ln\left(\frac{5}{3}\right)[\theta]_0^{2\pi}=4\pi\ln\left(\frac{5}{3}\right)$$

Example 1.25

Find the mass of the region corresponding to $x^2+y^2+z^2\leq 9,\ x\geq 0,\ y\geq 0$ if $\rho=xyz$

Solution

$$mass=\iiint_V \rho dxdydz=\iiint_V xyzdxdydz$$

$$x^2+y^2+z^2=9\Rightarrow r^2=9\Rightarrow r=3$$

$$m=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{3}(r\cos\theta\sin\phi)(r\sin\theta\sin\phi)(r\cos\phi)|J|drd\theta d\phi$$

$$=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{3}r^5\cos\theta\sin\theta\sin^3\phi\cos\phi drd\theta d\phi$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\left[\frac{r^6}{6}\right]_{r=0}^{3}\cos\theta\sin\theta\sin^3\phi\cos\phi d\theta d\phi$$

$$= \frac{243}{2}\int_{\theta=0}^{2\pi}\left[\frac{\sin^4\phi}{4}\right]_{\phi=0}^{\pi}\cos\theta\sin\theta d\theta = 0$$

Exercise 1.6

1. Evaluate

$$\iiint_T z dx dy dz$$

where T is the hemisphere of radius a, $x^2 + y^2 + z^2 = a^2$, $z \geq 0$

2. A solid fills the region between two concentric spheres of radii a and b $(0 < a < b)$. The density at each point is given by $\rho = \dfrac{k}{x^2 + y^2 + z^2}$, find the mass of the region

$$m = \iiint_T \rho dx dy dz$$

1.6 Line Integral

Definition

The line integral (over a curve C) denoted by

$$\oint_C (P dx + Q dy)$$

is an integral evaluated over a path (a curve) C where P and Q are continuous function on this path (See Fig. 1.15)

Let C be a curve in xy plane which connect points $A(a_1, b_1)$ and $B(a_2, b_2)$ (as shown above). Let $P(x, y)$ and $Q(x, y)$ be a single valued function define at all points of C (See Fig.1.16)

Let subdivide C into n parts, by choosing $(n - 1)$ points on it given by $(x_1, y_1), (x_2, y_2), \cdots, (x_{n-1}, y_{n-1})$. Let $\triangle x_i = x_i - x_{i-1}, \triangle y_i = y_i - y_{i-1}$ where $(a_1, b_1) = (x_0, y_0)$ and $(x_n, y_n) = (a_2, b_2)$ choose the point (ξ_i, η_i) such that this point is situated between (x_{i-1}, y_{i-1}) from the sum

$$\sum_{i=1}^{n}(P(\xi_i, \eta_i)\triangle x_i + Q(\xi_i, \eta_i)\triangle y_i)$$

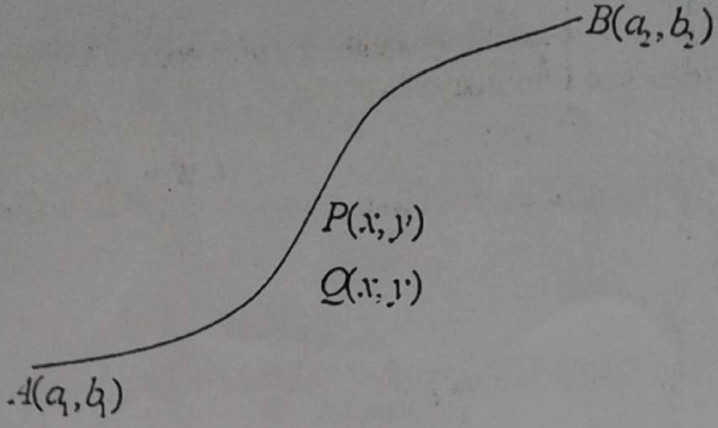

Figure 1.15:

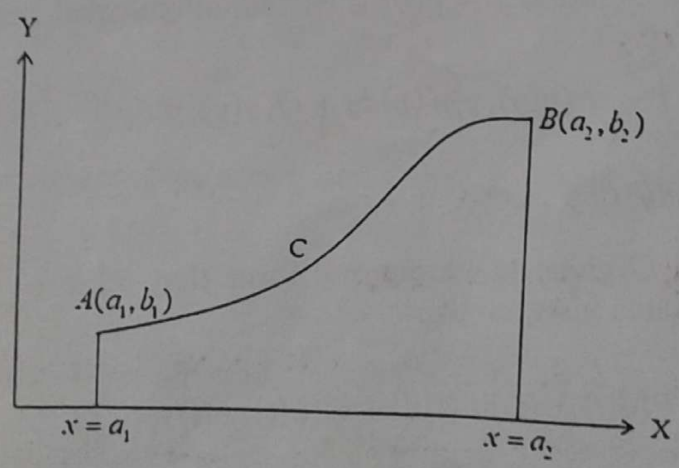

Figure 1.16:

The limit of the sum as $n \to \infty$ in such away that all the quantities $\Delta x_i, \Delta y_i$ approach zero is called the line integral along C that is,

$$\lim_{n\to\infty} \sum_{i=1}^{n} (P(\xi_i, \eta_i)\Delta x_i + Q(\xi_i, \eta_i)\Delta y_i) = \oint_C (Pdx + Qdy)$$

Note

$$\oint_C (Pdx + Qdy) = \int_{(a_1,b_1)}^{(a_2,b_2)} (Pdx + Qdy)$$

1.6.1 Evaluation of Line Integrals

1. If the curve C has equation $y = f(x)$ then the line integral becomes

$$\int_{x=a_1}^{a_2} P(x, f(x))dx + Q(x, f(x))f'(x)dx$$

   that is $dy = f'(x)dx$

2. If the curve C has equation $x = g(y)$ then the line integral becomes

$$\int_{y=b_1}^{b_2} P(g(y), y)g'(y)dy + Q(g(y), y)dy$$

   that is $dx = g'(y)dy$

3. item If the curve C given in parametric form that is, $x = \phi(t)$ and $y = \psi(t)$ then the line integral gives

$$\int_{t=t_1}^{t_2} P(\phi(t), \psi(t))\phi'(t)dt + Q(\phi(t), \psi(t))\psi'(t)dt$$

   that is $dx = \phi'(t)dt$, $dy = \psi'(t)dt$

Example 1.26 — exam question

Evaluate the line integral

$$I = \oint_C (x^2 + 2y)dx + (x + y)dy$$

from $A(0, 1)$ to $B(2, 3)$ along C defined by

(i) $y = x + 1$

(ii) along a straight line

(iii) along the parabola $x = t$ and $y = t^2 + 1$

(iv) along $(0, 1)$ to $(2, 1)$ then $(2, 3)$

Solution

(i)

$$\oint_C (x^2 + 2y) + (x + y)dy$$

$A(0, 1)$ and $B(2, 3)$ along the curve $y = x + 1$ then

$$y = x + 1 \Rightarrow dy = dx$$

$$\int_0^2 (x^2 + 4x + 3)dx = \left[\frac{x^3}{3} + 2x^2 + 3x\right]_0^2 = \left[\frac{8}{3} + 8 + 6\right] = \frac{8 + 42}{3} = \frac{50}{3}$$

(ii) Along the straight line $A(0, 1)$ to $B(2, 3)$

$$y - 1 = \frac{1 - 3}{0 - 2}(x - 0) \Rightarrow y - 1 = x \Rightarrow y = x + 1$$

as above.

(iii) Along the parabola $x = t$, and $y = t^2 + 1$ from $A(0, 1)$ to $B(2, 3)$

Q2b.cam

$$\oint_C (x^2 + 2y) + (x + y)dy$$

$$x = t \Rightarrow dx = dt, \quad y = t^2 + 1 \Rightarrow dy = 2tdt, \quad 0 \leq t \leq 1$$

$$\int_{t=0}^2 (t^2 + 2(t^2 + 1))dt + (t + t^2 + 1)2tdt = \int_0^2 (t^2 + 2t^2 + 2 + 2t^2 + 2t^3 + 2t)dt$$

$$= \int_0^2 (2t^3 + 5t^2 + 2t + 2)dt = \left[\frac{t^4}{2} + \frac{5t^3}{3} + t^2 + 2t\right]_0^2$$

$$= 8 + \frac{40}{3} + 4 + 4 = \frac{40 + 48}{3} = \frac{88}{3}$$

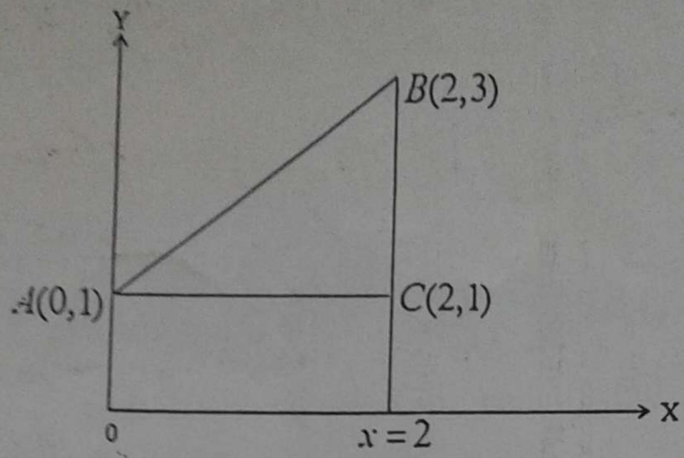

Figure 1.17:

(iv) Along the straight line $(0,1)$ to $(2,1)$ then to $(2,3)$ (See Fig. 1.17)
Along AC: $y = 1 \Rightarrow dy = 0$

$$\int_{x=0}^{2} (x^2+2)dx + (x+1)0 = \left[\frac{x^3}{3} + 2x\right]_0^2 = \frac{8}{3} + 4 = \frac{20}{3}$$

Along BC: $x = 2 \Rightarrow dx = 0$.

$$\int_1^3 (4+2y)0 + (2+y)dy = \left[2y + \frac{y^2}{2}\right]_1^3 = 6 + \frac{9}{2} - 2 - \frac{1}{2} = 4 + 4 = 8$$

$\therefore$ The line integral along $A(0,1)$ to $C(2,1)$ then to $B(2,3)$ is

$$\int_{along\ AC} + \int_{along\ CB} = \frac{20}{3} + 8 = \frac{44}{3}$$

Example 1.27

Evaluate

$$\int_{(1,1)}^{(4,2)} (x+y)dx + (y-x)dy$$

(i) along the parabola $x = y^2$

(ii) along straight line

(iii) from $(1,1)$ to $(1,2)$ then to $(4,2)$

(iv) the curve $x = 2t^2 + t + 1, y = t^2 + 1$

Solution

(i) Along $x = y^2 \Rightarrow dx = 2ydy$

$$\int_{y=1}^{2} (y^2 + y)2ydy + (y - y^2)dy = \int_{y=1}^{2} (2y^3 + 2y^2 + y - y^2)dy$$

$$\int_{1}^{2} (2y^3 + y^2 + y)dy = \left[\frac{y^4}{2} + \frac{y^3}{3} + \frac{y^2}{2}\right]_1^2 = (8 + \frac{8}{3} + 2) - (\frac{1}{2} + \frac{1}{3} + \frac{1}{2})$$

$$= 10 + \frac{8}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{2} = 9 + \frac{7}{3} = \frac{27+7}{3} = \frac{34}{3}$$

(ii) $\frac{y-1}{x-1} = \frac{2-1}{4-1} \Rightarrow y - 1 = \frac{1}{3}(x-1) \Rightarrow 3y - 3 = x - 1 \Rightarrow x = 3y - 2 \Rightarrow$

$dx = 3dy$

$$\int_{y=1}^{2} (3y - 2 + y)3dy + (y - (3y-2))dy = \int_{1}^{2} (12y - 6 - 2y + 2)dy$$

$$= \int_{1}^{2} (10y - 4)dy = \left[5y^2 - 4y\right]_1^2 = (20 - 8) - (5 - 4) = 12 - 1 = 11$$

(iii) Along AB: $x = 1 \Rightarrow dx = 0$

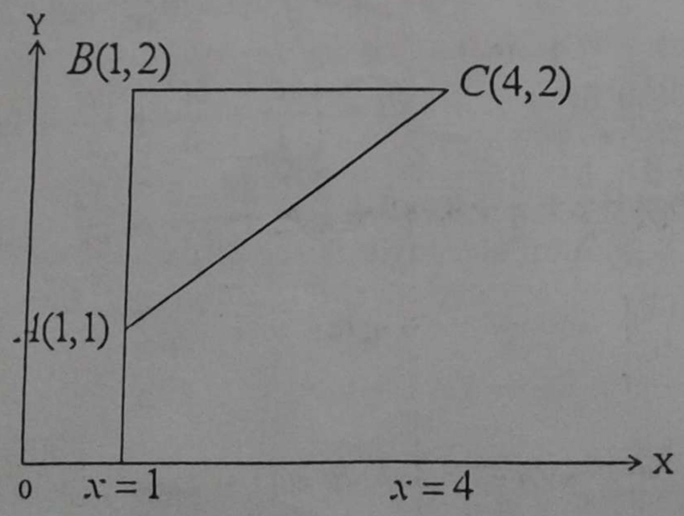

Figure 1.18:

$$\int_1^2 (1+y)0 + (y-1)dy = \left[\frac{y^2}{2} - y\right]_1^2 = (2-2) - \left(\frac{1}{2} - 1\right) = \frac{1}{2}$$

Along BC: $y = 2 \Rightarrow dy = 0$

$$\int_{x=1}^4 (x+2)dx + (2-x)0 = \left[\frac{x^2}{2} + 2x\right]_1^4 = 8 + 8 - \left(\frac{1}{2} + 2\right) = 16 - \frac{3}{2}$$

$$= \frac{32-3}{2} = \frac{29}{2}$$

$\therefore$ The line integral along $A(1,1)$ to $B(1,2)$ then to $C(4,2)$ is

$$\int_{along\ AB} + \int_{along\ BC} = \frac{1}{2} + \frac{29}{2} = \frac{30}{2} = 15$$

(iv) Along the curve $x = 2t^2 + t + 1,\ y = t^2 + 1 \Rightarrow dx = (4t+1)dt,\ dy = 2tdt$

$$\int_{t=1}^1 (2t^2 + t + 1 + t^2 + 1)(4t+1)dt + (t^2 + 1 - 2t^2 - t - 1)2tdt$$

$$= \int_0^1 [(3t^2+t+2)(4t+1)+(-t^2-t)2t]dt = \int_0^1 (12t^3+7t^2+9t+2-2t^3-2t)dt$$

$$= \int_0^1 (10t^3 + 5t^2 + 9t + 2)dt = \left[\frac{5t^4}{2} + \frac{5t^3}{3} + \frac{9t^2}{2} + 2t\right]_0^1$$

$$= \frac{5}{2} + \frac{5}{3} + \frac{9}{2} + 2 = 9 + \frac{5}{3} = \frac{27+5}{3} = \frac{32}{3}$$

Assignment

Example 1.28

Evaluate the integral

$$\oint_C (3x^2 - 6yz)dx + (2y + 3xz)dy + (1 - 4xyz^2)dz$$

from $(0,0,0)$ to $(1,1,1)$ along

(i) $x = t,\ y = t^2,\ z = t^3$

(ii) the straight line from $(0,0,0)$ to $(0,0,1)$ then to $(0,1,1)$ and then to $(1,1,1)$

(iii) the straight line joining $(0,0,0)$ to $(1,1,1)$

Solution

(i) $x = t \Rightarrow dx = dt,\ y = t^2 \Rightarrow dy = 2tdt,\ z = t^3 \Rightarrow dz = 3t^2dt$ and $0 \le t \le 1$

$$\int_{t=0}^{1} (3t^2 - 6(t^2)(t^3))dt + (2t^2 + 3(t)(t^3))2tdt + (1 - 4(t)(t^2)(t^3)^2)3t^2dt$$

$$= \int_0^1 (3t^2 - 6t^5 + 4t^3 + 6t^5 + 3t^2 - 12t^{11})dt = \int_0^1 (6t^2 + 4t^3 - 12t^{11})dt$$

$$= [2t^3 + t^4 - t^{12}]_0^1 = 2 + 1 - 1 = 2$$

(ii) Along the straight from $(0, 0, 0)$ to $(0, 0, 1)$. $x = 0,\ y = 0,\ dx = 0,\ dy = 0$ while z varies from 0 to 1.

$$\int_{z=0}^{1} (0 - 0)0 + (0 + 0 \cdot z)0 + (1 - 0 \cdot z)dz = \int_0^1 dz = [z]_0^1 = 1$$

Along the straight line from $(0, 0, 1)$ to $(0, 1, 1)$. $x = 0,\ z = 1,\ dx = 0,\ dz = 0$ while y varies from 0 to 1.

$$\int_{y=0}^{1} (0 - 6y)0 + (2y + 0)dy + (1 - 0)0 = \int_0^1 2ydy = [y^2]_0^1 = 1$$

Along the line from $(0, 1, 1)$ to $(1, 1, 1)$. $y = 1,\ z = 1,\ dy = 0,\ dz = 0$ while x varies from 0 to 1.

$$\int_{x=0}^{1} (3x^2 - 6)dx + (2 + 3x)0 + (1 - 4x)0$$

$$= \int_0^1 (3x^2 - 6)dx = [x^3 - 6x]_0^1 = 1 - 6 = -5$$

$\therefore$ The line integral along the straight line from $(0, 0, 0)$ to $(0, 0, 1)$ then to $(0, 1, 1)$ and then to $(1, 1, 1)$ is

$$\oint_C = 1 + 1 - 5 = -3$$

(iii) The straight line joining $(0, 0, 0)$ to $(1, 1, 1)$ is given by parametric equations $x = t,\ y = t,\ z = t,\ \Rightarrow dx = dt,\ dy = dt,\ dz = dt$ and $0 \le t \le 1$ then

$$\oint_C = \int_{t=0}^{1} (3t^2 - 6t^2)dt + (2t + 3t^2)dt + (1 - 4t^4)dt$$

$$= \int_0^1 (-3t^2 + 2t + 3t^2 + 1 - 4t^4)dt = \int_0^1 (1 + 2t - 4t^4)dt$$

$$= \left[t + t^2 - \frac{4t^5}{5}\right]_0^1 = 1 + 1 - \frac{4}{5} = \frac{10-4}{5} = \frac{6}{5}$$

Exercise 1.7

1. Evaluate the line integral

$$\oint_C x^2y\,dx + (x^2 - y^2)dy$$

from $(0,0)$ to $(1,4)$ where C is

   (a) the curve $y = 4x^2$
   (b) the line $y = 4x$

2. Evaluate the line integral

$$\oint_C (3x - 2y)dx + (y + 2x)dy - x^2dz$$

where C from $(0,0,0)$ to $(1,1,1)$ is the path consisting of

   (a) the curve $x = t,\ y = t^2,\ z = t^3$.
   (b) the straight line joining this points.
   (c) the straight line from $(0,0,0)$ to $(0,1,0)$ then to $(0,1,1)$ and then to $(1,1,1)$.
   (d) the curve $x = z^2,\ z = y^2$.

3. Evaluate

$$\oint_C (x - 3y)dx + (y - 2x)dy$$

where C is the curve $x = 2\cos t,\ y = 3\sin t$ from $t = 0$ to $t = 2\pi$

4. Evaluate

$$\oint_C (2x - y + 4)dx + (5y + 2x - 6)dy$$

around the triangle in the xy plane with vertices $(0,0), (3,0), (3,2)$ traversed in a counter clockwise direction.

1.6.2 Green's Theorem

Let R be a simple connected region bounded by a simple closed curve C (A closed curve C is said to be simple if it intersect itself at only one point) let the functions P, Q, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ be single-valued and continuous in the region R then

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Proof

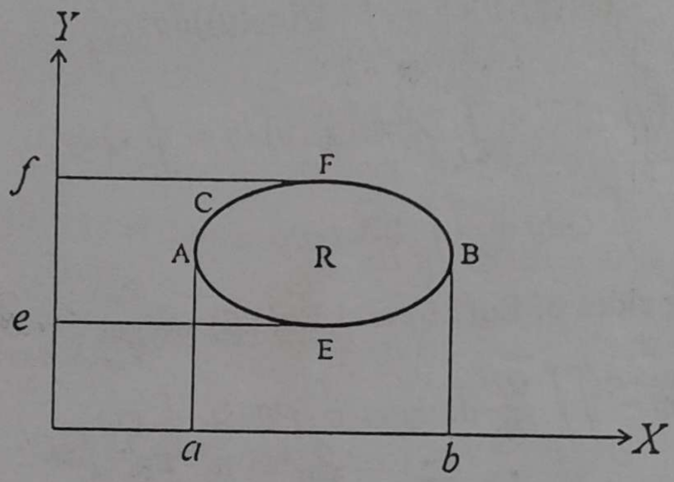

Figure 1.19:

Let the region bounded by AEB and BEA with the equations $y = y_1(x)$ and $y = y_2(x)$ respectively. Then

$$\int_a^b \left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial P}{\partial y} dy\right] dx = \int_{x=a}^b P(x,y)\Big|_{y_1(x)}^{y_2(x)} dx$$

$$= \int_a^b P(x, y_1(x))dx - \int_a^b P(x, y_2(x))dx$$

$$= -\int_b^a P(x, y_1(x))dx - \int_a^b P(x, y_2(x))dx = \oint_C Pdx$$

$$\oint_C P dx = -\iint_R \frac{\partial P}{\partial y} dxdy \tag{1.5}$$

Similarly: let the region R be bounded by the curves EBF and FAE with equations $x = x_1(y)$ and $x = x_2(y)$ respectively. Then

$$\int_{y=e}^{f} \left[\int_{x=x_1(y)}^{x=x_2(y)} \frac{\partial Q}{\partial x} dx \right] dy = \int_{y=e}^{f} Q(x,y) \Big|_{x_1(y)}^{x_2(y)} dy$$

$$= \int_e^f Q(x_1(y), y)dy - \int_e^f Q(x_2(y))dy$$

$$= \int_e^f P(x_1(y), y)dy + \int_f^e Q(x_2(y), y)dy = \oint_C Qdy$$

$$\oint_C Qdy = \iint_R \frac{\partial Q}{\partial x} dxdy \tag{1.6}$$

Adding the corresponding sides of Eq(1.5) and Eq(1.6) we get

$$\iint_R \frac{\partial Q}{\partial x} dxdy - \iint_R \frac{\partial P}{\partial y} dxdy = \oint_C Pdx + \oint_C Qdy$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_C (Pdx + Qdy).$$

exams question

Example 1.29

Verify Green's theorem in the xy plane for line integral.

$$\oint_C (2xy - x^2)dx + (x + y^2)dy$$

bounded by $y = x^2$ and $y^2 = x$

Solution

Using Green's theorem

$$P = 2xy - x^2,\ Q = x + y^2 \Rightarrow \frac{\partial P}{\partial y} = 2x,\ \frac{\partial Q}{\partial x} = 1$$

$$\oint_C (2xy - x^2)dx + (x + y^2)dy = \iint (1 - 2x)dxdy$$

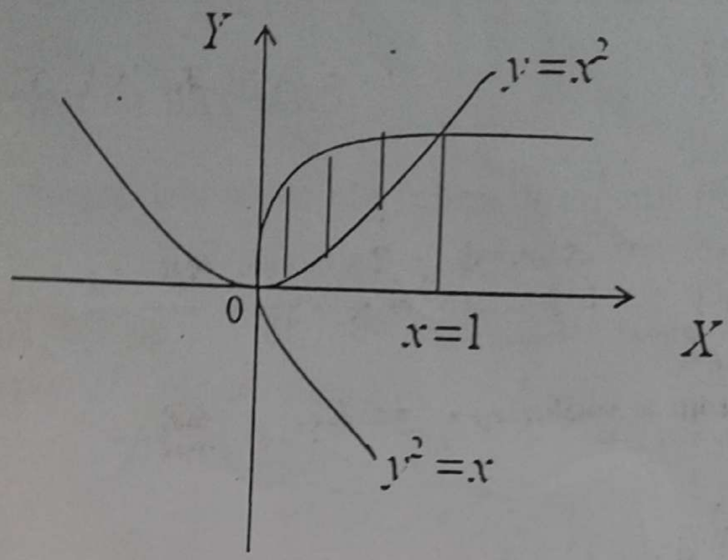

Figure 1.20:

But $x^4 = x \Rightarrow x(x^3-1) = 0 \Rightarrow x = 0$ *or* $x = 1$ when $x = 0\ y = 0,\ x = 1\ y = 1$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (1-2x)dydx = \int_{x=0}^{1} y - 2xy\Big|_{y=x^2}^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} (\sqrt{x} - 2x^{3/2} - x^2 + 2x^3)dx = \left[\frac{2x^{3/2}}{3} - \frac{4x^{5/2}}{5} - \frac{x^3}{3} + \frac{x^4}{2}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2} = \frac{20-24-10+15}{30} = \frac{1}{30}$$

Now Using line integral
Along $y = x^2$ we have $dy = 2xdx$

$$\oint_C (2xy - x^2)dx + (x+y^2)dy = \int_0^1 (2x^3 - x^2)dx + (x + x^4)2xdx$$

$$= \int_0^1 |(2x^3 - x^2 + 2x^2 + 2x^5)dx = \left[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{7}{6}$$

Along $x = y^2$ we have $dx = 2ydy$

$$\int_1^0 (2y^3 - y^4)2ydy + 2y^2dy = \int_1^0 (4y^4 - 2y^5 + 2y^2)dy$$

$$= \left[\frac{4y^5}{5} - \frac{y^6}{3} + \frac{2y^3}{3}\right]_1^0 = 0 - \left(\frac{4}{5} - \frac{1}{3} + \frac{2}{3}\right) = -\frac{12-5+10}{15} = -\frac{17}{15}$$

∴ the line integral is

$$\oint_C = \oint_{along\ y=x^2} + \oint_{along\ x=y^2} = \frac{7}{6} - \frac{17}{15} = \frac{35-34}{30} = \frac{1}{30}$$

Hence Green's theorem is verified.

Exercise 1.8

1. Verify Green's theorem in the xy plane for

$$\oint_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

where C is a square with vertices at $(0,0)(2,0)(2,2)$ and $(0,2)$.

2. Verify Green's theorem in the xy plane for

$$\oint_C (2x - y + 4)dx + (5y + 3x - 6)dy$$

where C is the triangle with the vertices at $(0,0)(3,0)(3,2)$.

3. (a) Let C be any simple closed curve bounded a region having Area A. Prove that if $a_1, a_2, a_3, b_1, b_2, b_3$ are constant,

$$\oint_C (a_1x + a_2y + a_3)dx + (b_1x + b_2y + b_3)dy = (b_1 - a_2)A$$

(b) Under what condition will the line integral around any path C be zero?

4. Verify Green's theorem in the xy plane for

$$\oint_C (x^3 - x^2y)dx + xy^2dy$$

where C is the boundary of the region enclosed by the circles $x^2+y^2=4$ and $x^2+y^2=16$

1.6.3 Line integral Independent of the Path

The line integral

$$\oint_C (Pdx + Qdy)$$

is said to be independent of the path taken if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\Rightarrow$ $\exists$ a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

That is,

$$P = \frac{\partial f}{\partial x}\ Q = \frac{\partial f}{\partial y}$$

in this case $F(x, y)$ is determined accordingly. The line integral can be Evaluated by definite integration.

$$\oint_C (Pdx + Qdy) = \int_{(a_1,b_1)}^{(a_2,b_2)} f(x, y)dxdy = F(x, y)\Big|_{(a_1,b_1)}^{(a_2,b_2)} = F(a_2, b_2) - F(a_1, b_1)$$

Example 1.30

Show that the line integral

$$\oint_{(1,2)}^{(3,4)} (6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy$$

is independent of the path taken and evaluate it.

Solution

$$P = 6xy^2 - y^3,\ Q = 6x^2y - 3xy^2$$
$$\frac{\partial P}{\partial y} = 12xy - 3y^2\ \frac{\partial q}{\partial y} = 12xy - 3y^2$$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ this implies that $\exists$ a scalar function $f(x, y) = c$ such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow P = \frac{\partial f}{\partial x} = 6xy^2 - y^3 \tag{1.7}$$

$$Q = \frac{\partial f}{\partial y} = 6x^2y - 3xy^2 \tag{1.8}$$

Integrating (1.7) with respect to x

$$f(x, y) = 3x^2y^2 - xy^3 + c(y) \tag{1.9}$$

Differentiating (1.9) with respect to y

$$\frac{\partial f}{\partial y} = 6x^2y - 3xy^2 - c'(y) \tag{1.10}$$

Comparing (1.10) and (1.8) we have

$$c'(y) = 0 \Rightarrow c(y) = C$$

$$f(x, y) = 3x^2y^2 - xy^3 + c$$

The integral

$$\oint_C Pdx + Qdy = \int_{(1,2)}^{(3,4)} (6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy = \left[3x^2y^2 - xy^3\right]_{(1,2)}^{(3,4)}$$

$$= [3(9)(16) - 3(64)) - (3(1)(4) - 1(8))]$$

$$= (432 - 192) - (12 - 8) = 240 - 4 = 236.$$

Alternatively, Evaluating the integral along the straight line joining the points $(1, 2)$ and $(3, 4)$ that is

$$y - 2 = \frac{2-4}{1-3}(x - 1) \Rightarrow y - 2 = x - 1 \Rightarrow y = x + 1 \Rightarrow dx = dy$$

$$\int_{x=1}^{3} [6x(x+1)^2 - (x+1)^3]dx + [6x^2(x+1) - (3x(x+1)^2]dx$$

$$= \int_{x=1}^{3} (6x^3 + 12x^2 + 6 - x^3 - 3x^2 - 3x - 1)dx + (6x^3 + 6x^2 - 3x^3 - 6x^2 - 3x)dx$$

$$= \int_1^3 (8x^3 + 9x^2 - 1)dx = 2x^4 + 3x^3 - x \,|_1^3$$

$$= [2(81) + 81 - 3] - (2 + 3 - 1) = 243 - 3 - 4 = 236$$

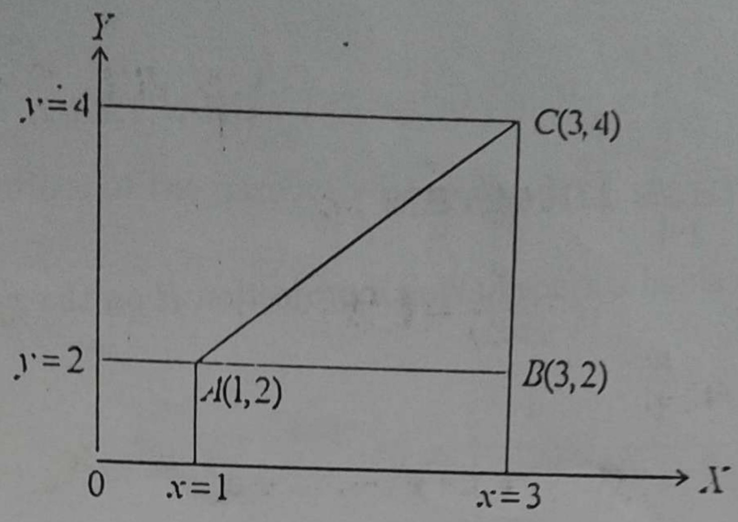

Figure 1.21:

Alternatively, let go along the straight line joining $(1,2)$ to $(3,2)$ then to $(3,4)$ (See Fig. 1.21)
Along AB: $y = 2 \Rightarrow dy = 0$ and x varies from 1 to 3

$$\int_{x=1}^{3} (24x - 8)dx = 12x^2 - 8x \,|_1^3 = (108 - 24) - (12 - 8) = 80$$

Along BC: $x = 3 \Rightarrow dx = 0$ and y varies from 2 to 4

$$\int_{y=2}^{4} (54y - 4y^2)dy = 27y^2 - 3y^3 \,|_2^4 = (432 - 192) - (108 - 24) = 156$$

$\therefore$ The line integral along the straight line joining $(1,2)$ to $(3,2)$ then to $(3,4)$ is

$$\oint_C = \oint_{along\ AB} + \oint_{along\ BC} = 80 + 156 = 236$$

Exercise 1.9

1. Prove that

$$\oint_{(1,0)}^{(2,1)} (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy$$

this integral is independent of the path from $(1,0)$ to $(2,1)$. Hence Evaluate the line integral along $(1,0)$ to $(1,1)$ then to $(2,1)$.

1.7 Surface Integrals

Let S be a two sided surface having a projection R on the xy plane (as shown below)

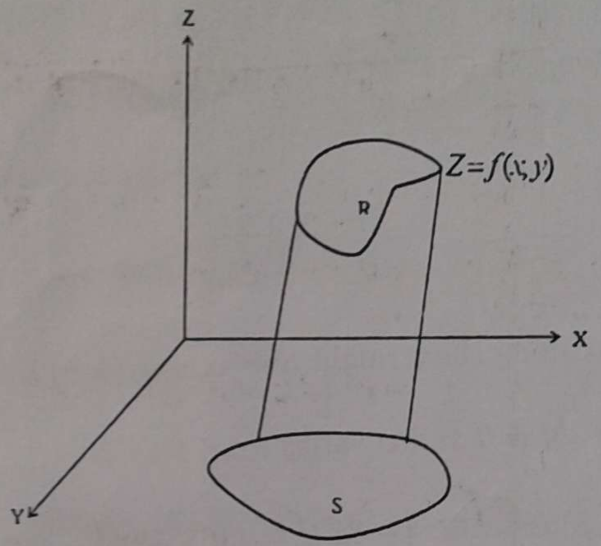

Figure 1.22:

Assume that the equation of the surface is given by $z = f(x, y)$ where f is single valued and continuous for all x and y in R. Let $\phi(x, y, z)$ be single valued and continuous at all point of S then if $z = f(x, y)$ has continuous derivatives in R, then the surface integral of $\phi(x, y, z)$ over S denoted by

$$\iint_S \phi(x, y, z)dS = \iint_R \phi(x, y, z)\sqrt{1 + z_x^2 + z_y^2}dxdy$$

If however the equation of the surface has the form $f(x, y, z) = 0$ then the surface integral becomes

$$\iint_S \phi(x, y, z)dS = \iint_R \phi(x, y, z)\frac{\sqrt{f_x^2 + f_y^2 + f_z^2}}{|f_z|}dxdy$$

the above stated result holds when $f(x, y, z) = 0$ is projected on the xy plane. Sometimes however, for convenience the ~~the~~ integral is projected on the xz plane or on the yz plane.
The surface integral define the flux(or density) of a quantity over the surface S if $\phi(x, y, z)$ is ρV when $\rho =$ density of the fluid and V is the velocity, then the surface integral gives the mass of the fluid passing over the surface S per unit time. e.t.c.

Example 1.31

Evaluate

$$\iint_S (x^2 + y^2)dS$$

where S is the surface of the paraboloid $z = 2 - (x^2 + y^2)$

Solution

$\phi = x^2 + y^2$

$$\iint_S (x^2 + y^2)dS = \iint_R (x^2 + y^2)\sqrt{1 + (-2x)^2 + (-2y)^2}dxdy$$

$$= \iint_R (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}dxdy$$

projecting $z = 2 - (x^2 + y^2)$ on the xy plane. $\Rightarrow z = 0 \Rightarrow 2 = x^2 + y^2$
Using the polar coordinate system

$$x^2 + y^2 = r^2 \Rightarrow 2 = r^2 \text{ or } r = \sqrt{2}$$

but $0 \leq r \leq \infty$

$$\int_{0=0}^{2\pi} \int_{r=0}^{\sqrt{2}} r^2\sqrt{1 + 4r^2} \cdot rdrd\theta$$

Let

$$u = 1 + 4r^2 \Rightarrow u - 1 = 4r^2 \Rightarrow \frac{1}{4}(u - 1) = r^2 \text{ and } du = 8rdr$$

$$I = \int \frac{1}{8} \cdot \frac{1}{4}(u - 1)u^{1/2}du = \frac{1}{32}\int (u^{3/2} - u^{1/2})du = \frac{1}{32}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)$$

$$= \frac{1}{32}\left[\frac{2}{5}(1 + 4r^2)^{5/2} - \frac{2}{3}(1 + 4r^2)^{3/2}\right]_0^{\sqrt{2}} = \frac{1}{32}\left[\left(\frac{2}{5} \cdot 3^5 - \frac{2}{3}3^3\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right]$$

$$= \frac{2}{32}\left[\frac{243}{5} - \frac{27}{3} - \frac{1}{5} + \frac{1}{3}\right] = \frac{1}{16}\left[\frac{242}{5} - \frac{26}{3}\right] = \frac{1}{8}\left[\frac{121}{5} - \frac{13}{3}\right]$$

$$= \frac{1}{8}\left[\frac{363 - 65}{15}\right] = \frac{1}{8} \times \frac{298}{15} = \frac{149}{60}$$

Therefore

$$\iint_S (x^2 + y^2)dS = \frac{149}{60}\int_{0=0}^{2\pi} d\theta = \frac{149}{60} \times 2\pi = \frac{149\pi}{30}$$

Example 1.32

Evaluate

$$\iint_S \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

where $\mathbf{A} = xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}$. S is the portion of the plane $2x + 2y + z = 6$ included in the first octant and $\hat{\mathbf{n}}$ is a unit normal to S.

Solution

A normal to S is $\nabla(2x + 2y + z - 6) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and so

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}$$

then

$$\mathbf{A} \cdot \hat{\mathbf{n}} = (xy\mathbf{i} - x^2\mathbf{j} + (x+z)\mathbf{k}) \cdot \left(\frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}\right) = \frac{2xy - 2x^2 + (x+z)}{3}$$

$$= \frac{2xy - 2x^2 + (x + 6 - 2x - 2y)}{3} = \frac{2xy - 2x^2 + -x - 2y + 6}{3}$$

the required surface integral is therefore

$$\iint_S \left(\frac{2xy - 2x^2 + -x - 2y + 6}{3}\right) dS$$

$$= \iint_R \left(\frac{2xy - 2x^2 + -x - 2y + 6}{3}\right) \sqrt{1 + z_x^2 + z_y^2} dx dy$$

$$= \iint_R \left(\frac{2xy - 2x^2 + -x - 2y + 6}{3}\right) \sqrt{1^2 + 2^2 + 2^2} dx dy$$

$$= \int_{x=0}^{3} \int_{y=0}^{3-x} (2xy - 2x^2 + -x - 2y + 6) dy dx$$

$$= \int_{x=0}^{3} [xy^2 - 2x^2y - xy - y^2 + 6y]_0^{3-x} dx$$

$$= \int_{x=0}^{3} [x(3-x)^2 - 2x^2(3-x) - x(3-x) - (3-x)^2 + 6(3-x)] dx$$

$$= \int_{x=0}^{3} (9x - 6x^2 + x^3 - 6x^2 + 2x^3 - 3x + x^2 - 9 + 6x - x^2 + 18 - 6x) dx$$

$$= \int_0^3 (3x^3 - 12x^2 + 6x + 9)dx = \left[\frac{3x^4}{4} - 4x^3 + 3x^2 + 9x\right]_0^3$$

$$= \frac{243}{4} - 108 + 27 + 27 = \frac{243}{4} - 54 = \frac{243 - 216}{4} = \frac{27}{4}$$

Exercise 1.10

1. Evaluate

$$\iint_S (x^2 + y^2)dS$$

where S is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $x = 3$.

2. Determine the surface area of the plane $2x + y + 2z = 16$ cut off by

   (a) $x = 0,\ y = 0,\ x = 2,\ y = 3$

   (b) $x = 0,\ y = 0$ and $x^2 + y^2 = 64$

3. Evaluate

$$\iint_S 3z dS$$

where S is the surface of the paraboloid $z = 2 - (x^2 + y^2)$ above the xy plane.

1.7.1 Divergence Theorem

Let S be a closed surface which is such that any line parallel to the coordinates axis cuts S in at most two point

$$\iint_S \mathbf{r} \cdot \hat{\mathbf{n}} dS = \iiint_V \nabla \cdot \mathbf{r} dV$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the surface

Example 1.33

Verify the divergence theorem for $\mathbf{A} = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}$ taken over the region bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Solution

We first evaluate

$$\iint_S \mathbf{A}\cdot\hat{\mathbf{n}}dS$$

where S is the surface of the cube as shown below

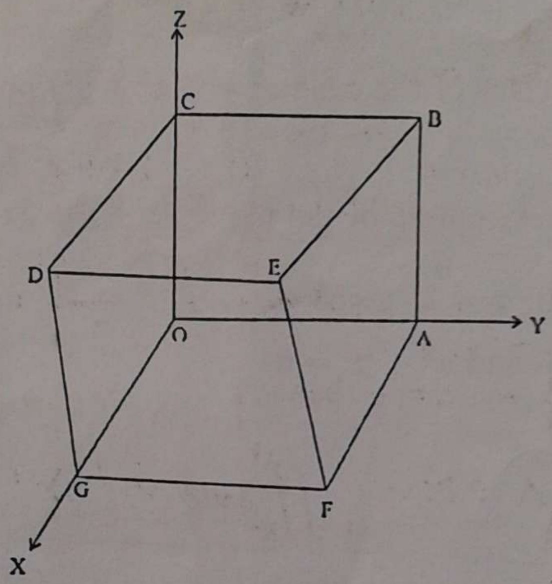

Figure 1.23:

for face DEFG $\hat{\mathbf{n}}=\mathbf{i},\ x=1$ then

$$\iint_{DEFG}\mathbf{A}\cdot\hat{\mathbf{n}}dS=\int_{y=0}^{1}\int_{z=0}^{1}((2-z)\mathbf{i}+\mathbf{j}-z^2\mathbf{k})\cdot\mathbf{i}dydz=\int_{y=0}^{1}\int_{z=0}^{1}(2-z)dzdy$$

$$=\int_{y=0}^{1}\left[2z-\frac{z^2}{2}\right]_0^1dy=\frac{3}{2}\int_0^1dy=\frac{3}{2}[y]_0^1=\frac{3}{2}$$

for face ABCO $\hat{\mathbf{n}}=-\mathbf{i},\ x=0$ then

$$\iint_{ABCO}\mathbf{A}\cdot\hat{\mathbf{n}}dS=\int_{y=0}^{1}\int_{z=0}^{1}(-z\mathbf{i})\cdot(-\mathbf{i})dydz=\int_{y=0}^{1}\int_{z=0}^{1}zdzdy$$

$$=\int_0^1\left[\frac{z^2}{2}\right]_0^1dy=\frac{1}{2}\int_0^1dy=\frac{1}{2}$$

for face ABEF $\hat{n} = j$, $y = 1$ then

$$\iint_{ABEF} A \cdot \hat{n} dS = \int_{x=0}^{1}\int_{z=0}^{1} ((2x-z)i + x^2 j - xz^2 k) \cdot j dx dz = \int_{z=0}^{1}\int_{x=0}^{1} x^2 dx dz$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_0^1 dz = \frac{1}{3}\int_0^1 dz = \frac{1}{3}$$

for face CDGO $\hat{n} = -j$, $y = 0$ then

$$\iint_{CDGO} A \cdot \hat{n} dS = \int_{x=0}^{1}\int_{z=0}^{1} ((2x-z)i - xz^2 k) \cdot (-j) dx dz = 0$$

for face BCDE $\hat{n} = k$, $z = 1$ then

$$\iint_{BCDE} A \cdot \hat{n} dS = \int_{x=0}^{1}\int_{y=0}^{1} ((2x-1)i + x^2 yj - xk) \cdot k dx dy$$

$$\int_{x=0}^{1}\int_{y=0}^{1} (-x) dx dy = \int_0^1 \left[\frac{-x^2}{2}\right]_0^1 = -\frac{1}{2}$$

for face AFGO $\hat{n} = -k$, $z = 0$ then

$$\iint_{AFGO} A \cdot \hat{n} dS = \int_{x=0}^{1}\int_{y=0}^{1} (2xi + x^2 yj) \cdot (-k) dx dy = 0$$

Adding we have

$$\iint_S A \cdot \hat{n} dS = \frac{3}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{2} + 0 = \frac{9+3+2-3}{6} = \frac{11}{6}$$

Now

$$\iiint_V \nabla \cdot A dV = \int_0^1\int_0^1\int_0^1 (2 + x^2 - 2xz) dx dy dz$$

$$= \int_0^1\int_0^1 \left[2x + \frac{x^3}{3} - x^2 z\right]_0^1 dy dz = \int_0^1\int_0^1 \left[2 + \frac{1}{3} - z\right] dy dz$$

$$= \int_0^1\int_0^1 \left[\frac{7}{3} - z\right] dy dz = \int_0^1 \left[\frac{7y}{3} - yz\right]_0^1 dz$$

$$= \int_0^1 \left[\frac{7}{3} - z\right] dz = \left[\frac{7z}{3} - \frac{z^2}{2}\right]_0^1 dz = \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \frac{11}{6}$$

Hence the divergence theorem if verified in this case.

Example 1.34

Verify the divergence theorem for $\mathbf{A} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4,\ z = 0$ and $z = 3$.

Solution

Volume integral

$$\iiint_V \nabla \cdot \mathbf{A} dV = \iiint_V (4 - 4y + 2z) dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z) dz dy dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4z - 4yz + z^2]_0^3 dy dx$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) dy dx$$

$$= \int_{x=-2}^{2} [21y - 6y^2]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} [[21\sqrt{4-x^2} - 6(4-x^2)] - [-21\sqrt{4-x^2} - 6(4-x^2)]] dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2} dx$$

Let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta;\ x = 2 \Rightarrow 2 = 2\sin\theta \Rightarrow \theta = \frac{\pi}{2};$
$x = -2 \Rightarrow -2 = 2\sin\theta \Rightarrow \theta = -\frac{\pi}{2}$

$$= 42 \int_{-\pi/2}^{\pi/2} \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta d\theta = 168 \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = 84 \int_{-\pi/2}^{\pi/2} (\cos 2\theta + 1) d\theta$$

$$= 84 \left[\frac{\sin 2\theta}{2} + \theta\right]_{-\pi/2}^{\pi/2} = 84[(0 + \pi/2) - (0 - \pi/2) = 84\pi$$

The surface S of the cylinder of base $S_1(z = 0)$ and top $S_2(z = 3)$ and the convex portion $S_3(x^2 + y^2 = 4)$ then the surface integral

$$\iint_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \iint_{S_1} \mathbf{A} \cdot \hat{\mathbf{n}} dS + \iint_{S_2} \mathbf{A} \cdot \hat{\mathbf{n}} dS + \iint_{S_3} \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

On $S_1(z=0)$ $\hat{n} = -k$, $A = 4xi - 2y^2j$ and $A \cdot \hat{n} = 0 \Rightarrow \iint_{S_1} A \cdot \hat{n} dS_1 = 0$

On $S_2(z=3)$ $\hat{n} = k$, $A = 4xi - 2y^2j + 9k$ and $A \cdot \hat{n} = 9 \Rightarrow \iint_{S_2} A \cdot \hat{n} dS_2 = 9\iint_{S_2} dS_2 = 36\pi$ since the area of S_2 is 4π.

On $S_3(x^2+y^2=4)$ a perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2xi + 2yj$ then the unit normal is

$$\hat{n} = \frac{2xi + 2yj}{\sqrt{4x^2+4y^2}} = \frac{xi + yj}{\sqrt{x^2+y^2}} = \frac{xi+yj}{2}, \textit{ since } x^2+y^2 = 4$$

$$A \cdot \hat{n} = (4xi - 2y^2j + z^2k) \cdot \left(\frac{xi+yj}{2}\right) = 2x^2 - y^3$$

Using cylindrical polar coordinates $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2dzd\theta$

$$\iint_{S_3} A \cdot \hat{n} dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3] 2dzd\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{3} (16\cos^2\theta - 16\sin^3\theta)dzd\theta = 48\int_{\theta=0}^{2\pi}(\cos^2\theta - \sin^3\theta)$$

$$= 48\int_{\theta=0}^{2\pi}\left[\frac{1}{2}(\cos 2\theta + 1) - \sin\theta(1-\cos^2\theta)\right] d\theta$$

$$= 48\int_{\theta=0}^{2\pi}\left[\frac{1}{2}(\cos 2\theta + 1) - \sin\theta - \sin\theta\cos^2\theta)\right] d\theta$$

$$= 48\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2} + \cos\theta - \frac{\cos^3\theta)}{3}\right]_0^{2\pi}$$

$$= 48[(0 + \pi + 1 - 1/3) - (0 + 0 + 1 - 1/3)] = 48\pi$$

The surface integral

$$\iint_S A \cdot \hat{n} dS = \iint_{S_1} A \cdot \hat{n} dS_1 \iint_{S_2} A \cdot \hat{n} dS_2 + \iint_{S_3} A \cdot \hat{n} dS_3 = 0 + 36\pi + 48\pi = 84\pi$$

Hence the divergence theorem is satisfied.

Exercise 1.11

1. evaluate

$$\iint_S \mathbf{r}\cdot\hat{\mathbf{n}}dS$$

   where S is a closed surface.

2. Verify the divergence theorem for $\mathbf{A} = (2xy+z)\mathbf{i}+y^2\mathbf{j}-(x+3y)\mathbf{k}$ taken over the region bounded by $2x+2y+z=6,\ x=0,\ y=0,\ z=0$.

3. Determine the value of

$$\iint_S xdydz + ydzdx + zdxdy$$

   where S is the surface of the region bounded by the $x^2+y^2=9$ and the plane $z=0$ and $z=3$

   (a) by using divergence theorem,

   (b) directly.

1.7.2 Stroke's Theorem

States that, let S be a surface which is such that its projection on the xy plane are region bounded by simple closed curves and assuming $z=f(x,y,z)$ *or* $x=g(y,z)$ *or* $y=h(x,y)$ where f,g,h are single-valued, continuous and differentiable functions then

$$\iint_V (\nabla\times\mathbf{A})\cdot\hat{\mathbf{n}}dS = \oint_C \mathbf{A}\cdot d\mathbf{r}$$

Exercise 1.35

Verify Stroke's theorem for $\mathbf{A} = 3y\mathbf{i} - xz\mathbf{j} + yz^2\mathbf{k}$ where S is the surface of the paraboloid $2z = x^2+y^2$ bounded by $z=2$ and C is its boundary.

Solution

The boundary C of S is a circle with equation $x^2+y^2=4,\ z=2$ and parametric equations $x=2\cos t,\ y=2\sin t,\ z=2 \Rightarrow dx=-2\sin t dt,\ dy=2\cos t dt,\ dz=0$ where $0\le t\le 2\pi$ then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C 3ydx - xzdy + yz^2dz.$$

$$= \int_{2\pi}^{0} 3(2\sin t)(-2\sin t)dt - (2\cos t)(2)(2\cos t)dt$$

$$= \int_0^{2\pi} (12\sin^2 t + 8\cos^2 t)dt = \int_o^{2\pi} (8 + 4\sin^2 t)dt$$

$$= \int_0^{2\pi} (8 + 2(1 - \cos 2t))dt = \int_0^{2\pi} (10 - 2\cos 2t)dt$$

$$= [-\sin 2t + 10t]_0^{2\pi} = 20\pi$$

Also

$$\nabla \times \mathrm{A} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= \mathrm{i}(z^2 + x) - \mathrm{j}(0 - 0) + \mathrm{k}(-z - 3) = (z^2 + x)\mathrm{i} - (z + 3)\mathrm{k}$$

$$\hat{\mathrm{n}} = \frac{\nabla(x^2 + y^2 - 2z)}{|\nabla(x^2 + y^2 - 2z)|} = \frac{2x\mathrm{i} + 2y\mathrm{j} - 2\mathrm{k}}{\sqrt{(2x)^2 + (2y)^2 + (-2)^2}} = \frac{x\mathrm{i} + y\mathrm{j} - \mathrm{k}}{\sqrt{x^2 + y^2 + 1}}$$

$$\iint_S (\nabla \times \mathrm{A}) \cdot \hat{\mathrm{n}} dS = \iint_S (\nabla \times \mathrm{A}) \cdot \hat{\mathrm{n}} \frac{dxdy}{|\hat{\mathrm{n}} \cdot \mathrm{k}|} = \iint_R (xz^2 + x^2 + z + 3)dxdy$$

$$= \iint_R \left[x\left(\frac{x^2 + y^2}{2}\right)^2 + x^2 + \frac{x^2 + y^2}{2} + 3 \right] dxdy$$

Using polar coordinates we have

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left[(r\cos\theta)\frac{r^4}{4} + r^2\cos^2\theta + \frac{r^2}{2} + 3 \right] rdrd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left[\frac{r^6}{4}\cos\theta + r^3\cos^2\theta + \frac{r^3}{2} + 3r \right] drd\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^7}{24}\cos\theta + \frac{r^4}{4}\cos^2\theta + \frac{r^4}{8} + \frac{3r^2}{2} \right]_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{16}{3}\cos\theta + 4\cos^2\theta + 2 + 6 \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{16}{3}\cos\theta + 2(\cos 2\theta + 1) + 8 \right] d\theta$$

$$= \left[\frac{16}{3}\sin\theta + \sin 2\theta + 2\theta + 8\theta\right]_{\theta=0}^{2\pi}$$

$$= [0 + 0 + 2(2\pi) + 8(2\pi)] = 4\pi + 16\pi = 20\pi$$

Exercise 1.1.12

1. Prove that the necessary and sufficient condition that

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$

for every closed curve C is that $\nabla \cdot \mathbf{A} = 0$ identically.

2. Verify Stoke's theorem for $\mathbf{A} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$ and C is its boundary.

3. Verify Stoke's theorem for $\mathbf{A} = (y + z)\mathbf{i} - xz\mathbf{j} + y^2\mathbf{k}$, where S is the surface of the region in the first octant bounded by $2x + z = 6$ and $y = 2$ which is not included in the

   (a) xy plane

   (b) plane $y = 2$

   (c) plane $2x + z = 6$ and C is the corresponding boundary.

Chapter 2

Fourier Series

Definition

A function is said to be periodic if its graph is repeated after a length of interval p that is, $f(x+p) = f(x), \forall x$ in the domain of f, p is called the period, for example $\sin x$, $\cos x$ e.t.c are periodic functions

$$f(x) = \begin{cases} 3 & 0 < x < 1 \\ -3 & -1 < x < 0 \end{cases}$$

2.1 Fourier Series

The Fourier series representation of a function (periodic) $f(x)$ is a trigonometric series of the form

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The above is a representation of $f(x)$ in terms of simple periodic function $\sin x$ or $\cos x$. The series has a very important application in science and Engineering especially in mechanic and Heat flow.
Note the following

1. $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

2.
$$\int_{-\pi}^{\pi} \cos mx dx = 0$$

that is,
$$\left[\frac{\sin mx}{m}\right]_{-\pi}^{\pi} = \frac{1}{m}(\sin m\pi - \sin -m\pi) = 0$$

3.
$$\int_{-\pi}^{\pi} \sin mx dx = \frac{1}{m}[-\cos mx]_{-\pi}^{\pi} = 0$$

4.
$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x + \sin(m-n)x]dx$$
$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

Suppose $m \neq n$
$$= -\frac{1}{2}\left[\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right.$$
$$\left. - \left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right)\right] = 0$$

5.
$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x + \cos(m-n)x]dx$$
$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

Suppose $m \neq n$
$$= \frac{1}{2}\left[\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n}\right.$$
$$\left. - \left(\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n}\right)\right] = 0$$

If $m = n$ in (5)
$$\int_{-\pi}^{\pi} \cos^2 mx dx = \frac{1}{2}\int_{-\pi}^{\pi}(1 + \cos 2mx)dx$$
$$= \frac{1}{2}\left[x + \frac{\sin 2mx}{2m}\right]_{-\pi}^{\pi} = \frac{1}{2}[\pi + 0 - 9 - \pi + 0)] = \pi$$

6.

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x) dx = 0$$

7. If $m = n$ in (6)

$$\int_{-\pi}^{\pi} \sin 2mx dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2mx) dx = \pi$$

Therefore

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & if\ m \neq n \\ \pi & if\ m = n \end{cases}$$

2.1.1 Computation of Fourier Coefficients

If

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

then the coefficients a_0, a_n, and b_n are called Fourier coefficient given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$$

Proof

Integrate equation (2.1) from $-\pi$ to π

$$\Rightarrow \int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)dx = a_0 \cdot 2\pi \ \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

or

$$\frac{a_o}{2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

To determine a_n multiply equation (2.1) by $\cos nx$ and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos^2 nx dx + b_n \int_{-\pi}^{\pi} \cos nx \sin nx dx \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cos nx dx = a_n \cdot \pi \Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx.$$

To determine b_n multiply equation (2.1) by $\sin nx$ and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= a_0 \int_{-\pi}^{\pi} \sin nx dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \sin nx \cos nx dx + b_n \int_{-\pi}^{\pi} \sin^2 nx dx \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \sin nx dx = b_n \cdot \pi \Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Hence from the above we can be express $f(x)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $f(x)$ is periodic in $[-\pi, \pi]$ and

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \; a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \; b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Example 2.1

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$$

Solution

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{0} x dx + \frac{1}{\pi} \int_{0}^{\pi} 0 dx$$

$$\frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{0} = \frac{1}{\pi} \left[0 - \frac{\pi^2}{2} \right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{0} x \cos nx dx + \frac{1}{\pi} \int_{0}^{\pi} 0 dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\Big|_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1-\cos n\pi}{n^2}\right]$$

$$a_n = \begin{cases} 0, & If\ n\ is\ even \\ \frac{2}{n^2\pi}, & If\ n\ is\ odd \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} 0 dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\Big|_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi\cos n\pi}{n}\right] = \frac{\cos n\pi}{n}$$

$$b_n = \frac{(-1)^n}{n}$$

$$\therefore f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\cos nx + \frac{(-1)^n}{n}\sin nx\right)$$

Or

$$f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + \left(-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \cdots\right)$$

2.1.2 Odd and even Functions

Definition

A function $f(x)$ is said to be an even function if $f(-x) = f(x), \forall x$ in the interval e.g $\cos x$, x^2, $\frac{x^4+3x^2}{x^2+1}$ e.t.c.

A function $f(x)$ is called an odd function if $f(-x) = -f(x), \forall x$ in the interval e.g $\sin x$, x, x^3 e.t.c.

Note: If $f(x)$ is an odd function, then $a_0 = a_n = 0$ the Fourier series has only sine terms present that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Similarly if $f(x)$ is an even function, then $b_n = 0$ and the Fourier series posses only the constant $\frac{a_o}{2}$ and the cosine terms that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Proof

Let $f(x)$ be an even function then $f(x) = f(-x)$ and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx - b_n \sin nx)$$

Therefore $f(x) = f(-x)$ if and only if

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx - \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow 2\sum_{n=1}^{\infty} b_n \sin nx = 0 \Rightarrow b_n = 0$$

If $f(x)$ is odd $\Rightarrow f(-x) = -f(x)$ if and only if

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx - \sum_{n=1}^{\infty} b_n \sin nx = -\frac{a_0}{2} - \sum_{n=1}^{\infty} a_n \cos nx - \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow a_0 + 2\sum_{n=1}^{\infty} b_n \cos nx = 0 \Rightarrow a_0 = a_n = 0$$

Example 2.2

Find the Fourier series expansion of $f(x) = x$ on the interval $[-\pi, \pi]$.

Solution

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin nx}{n}dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nx}{n}dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\right]_{-\pi}^{\pi} + 0$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right] = -\frac{2\cos n\pi}{n}$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = 2\sum_{n=1}^{\inf}\frac{(-1)^{n+1}}{n}\sin nx = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

2.1.3 Change of period from 2π to $2l$

If the period of the function $f(x)$ define with the interval $[-\pi, \pi]$ change to $2l$ defined by $[-l, l]$ then the Fourier series expansion of $f(x)$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\pi/l)x + b_n\sin(n\pi/l)x)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx,\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(n\pi/l)x dx,\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(n\pi/l)x dx$$

Example 2.3

Find the Fourier series expansion of the function

$$f(x) = \begin{cases} -3, & 0 \le x \le 2 \\ 3, & -2 \le x \le 0 \end{cases}$$

Solution

The period is 4 because it lies between -2, to 2.

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} 3dx + \frac{1}{2}\int_{0}^{2} -3dx$$

$$= \frac{1}{2}[3x]_{-2}^{0} + \frac{1}{2}[-3x]_{0}^{2} = \frac{1}{2}[0+6] + \frac{1}{2}[-6+0] = 0$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos(n\pi/2)xdx$$

$$= \frac{1}{2}\int_{-2}^{0} 3\cos(n\pi/2)xdx + \frac{1}{2}\int_{0}^{2} -3\cos(n\pi)xdx$$

$$= \frac{3}{n\pi}[\sin(n\pi/2)x]_{-2}^{0} - \frac{3}{n\pi}[\sin(n\pi/2)x]_{0}^{2}$$

$$= \frac{3}{n\pi}[0 - \sin n\pi] - \frac{3}{n\pi}[\sin n\pi - 0] = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin(n\pi/2)xdx$$

$$= \frac{1}{2}\int_{-2}^{0} 3\sin(n\pi/2)xdx + \frac{1}{2}\int_{0}^{2} -3\sin(n\pi)xdx$$

$$= -\frac{3}{n\pi}[\cos(n\pi/2)x]_{-2}^{0} + \frac{3}{n\pi}[\cos(n\pi/2)x]_{0}^{2}$$

$$= \frac{-3}{n\pi}[1 - \cos n\pi] + \frac{3}{n\pi}[\cos n\pi - 1]$$

$$= \frac{-6}{n\pi}[1 - \cos n\pi]$$

$$b_n = \begin{cases} \frac{-12}{n\pi}, & If\ n\ is\ odd \\ 0, & If\ n\ is\ even \end{cases}$$

Therefore

$$f(x) = \sum_{n=1}^{\text{inf}} \frac{-12}{n\pi} \sin(n\pi/2)x \quad for\ odd\ n$$

that is

$$f(x) = \frac{-12}{n\pi}\left[\frac{\sin(\pi x/2)}{1} + \frac{\sin(3\pi x/2)}{3} + \frac{\sin(5\pi x/2)}{5} + \cdots\right]$$

2.1.4 A half range Fourier sine or cosine Series

If a function is define in the half interval $[0, \pi]$ is a series which contains only the sine or cosine terms of the series that is,

$$f(x) = \frac{a_0}{2} + a_1 \cos n_1 x + a_2 \cos n_2 x + \cdots$$

or

$$f(x) = b_1 \sin n_1 x + b_2 \sin n_2 x + b_3 \sin n_3 x + \cdots$$

In this case the Fourier coefficients $\frac{a_0}{2}$, a_n and b_n have the form

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx,\ a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx,\ b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx$$

Example 2.4

Find the Fourier cosine series expansion of $f(x) = x^2$, $0 \le x \le \pi$.

Solution

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx = \frac{2}{\pi}\int_0^\pi x^2 \cos nx dx$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2 \sin nx}{n}\right)_0^\pi - \frac{2}{n}\int_0^\pi x \sin nx dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2 \sin nx}{n}\right)_0^\pi - \frac{2}{n}\left[\left(-\frac{x\cos nx}{n}\right)_0^\pi + \frac{1}{n}\int_0^\pi \cos nx dx\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^\pi$$

Fourier sine series → b_n
Fourier cosine series a_0 and a_n

$$= \frac{2}{\pi}\left[0 + \frac{2\pi\cos n\pi}{n^2} - 0 - (0 + 0 - 0)\right] = \frac{4\cos n\pi}{n^2}$$

$$a_n = \frac{4(-1)^n}{n^2} \quad \forall n$$

Therefore

$$f(x) = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx = \frac{2\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^3} + \cdots\right].$$

Example 2.5 — b_n

Find the Fourier sine series of $f(x) = \pi - x,\ 0 \le x \le 3$.

Solution

$$b_n = \frac{2}{l}\int_0^l f(x)\sin(n\pi/l)x\,dx$$

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin(n\pi/3)x\,dx$$

$$b_n = \frac{2}{3}\left[\left(-(\pi - x)\cdot\frac{3}{n\pi}\cos(n\pi/3)x\right)_0^3 - \frac{3}{n\pi}\int_0^3 \cos(n\pi/3)x\,dx\right]$$

$$b_n = \frac{2}{3}\left[\frac{-3(\pi - x)\cos(n\pi/3)x}{n\pi} - \frac{9}{n^2\pi^2}\sin(n\pi/3)x\right]_0^3$$

$$b_n = \frac{2}{3}\left[\frac{-3(\pi - 3)\cos n\pi}{n\pi} - 0 - \left(\frac{-3\pi}{n\pi} - 0\right)\right]$$

$$= \frac{2}{3}\left[\frac{-3}{n} + \frac{9\cos n\pi}{n\pi} + \frac{3}{n}\right]$$

$$b_n = \frac{6\cos n\pi}{n\pi} = \frac{6(-1)^n}{n\pi}$$

Therefore

$$f(x) = \frac{6}{\pi}\left[-\frac{\sin(\pi x/3)}{1} + \frac{\sin(2\pi x/3}{2} - \frac{\sin \pi x}{3} + \cdots\right]$$

Exercise 2.1

1. Find the Fourier series expansion of

   (a)

$$f(x) = \begin{cases} \pi - x, & If\ -\pi \le x \le 0 \\ x, & If\ o \le x \le \pi \end{cases}$$

   (b) $f(x) = \frac{x^2}{2}, -4 \le x \le 4$

2. Find the half range Fourier sine and cosine series of

   (a) $f(x) = 1 - x,\ 0 < x < 3$

   (b) $f(x) = 1 - x^3,\ 0 < x < 6$

3. Expand

$$f(x) = \begin{cases} 2 - x, & 0 < x < 4 \\ x - 6, & 4 < x < 8 \end{cases}$$

   in Fourier sense if period is 8.

Chapter 3

Laplace Transforms

Definition

The Laplace transform of a function $f(t)$, where $0 \le t < \infty$ denoted by

$$F(s) = \mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)dt, \; s > 0.$$
$$= \lim_{m\to\infty} \int_0^m e^{-st} f(t)dt.$$

The Laplace transformed of $f(t)$ exist if the above integral converges, other wise the Laplace transformed does not exist.

3.1 Evaluation of Laplace Transform of Functions

Example 3.1

Find the Laplace transformed of e^{at}

Solution

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt$$
$$= \lim_{m\to\infty} \int_0^m e^{-(s-a)t} dt = \lim_{m\to\infty} \left[\frac{-1}{s-a} e^{-(s-a)t} \right]_0^m$$
$$= \lim_{m\to\infty} \left[\frac{1}{s-a} - \frac{e^{-(s-a)m}}{s-a} \right] = \frac{1}{s-a}$$

Therefore

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}.$$

Example 3.2

Compute the Laplace transform of $\mathcal{L}\{(1/t)\}$

Solution

$$\mathcal{L}\{(1/t)\} = \int_0^\infty e^{-st}\frac{1}{t} = \infty \; (diverges)$$

That is the limit does not exist.

Example 3.3

Find the Laplace transform of $\{\sin at\}$ and $\{\cos at\}$.

Solution

Note that

$$e^{iat} = \cos(at) + i\sin(at)$$

which implies that

$$\mathcal{L}\{\cos(at)\} = Re\mathcal{L}\{e^{iat}\}$$

and

$$\mathcal{L}\{\sin(at)\} = Im\mathcal{L}\{e^{iat}\}$$

Now

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st}\cdot e^{iat}dt = \int_0^\infty e^{-(s-ia)t}dt$$

$$= \lim_{m\to\infty}\int_0^m e^{-(s-ia)t}dt = \lim_{m\to\infty}\left[\frac{-1}{s-ia}e^{-(s-ia)t}\right]_0^m$$

$$= \lim_{m\to\infty}\left[\frac{1}{s-ia} - \frac{e^{-(s-ia)m}}{s-ia}\right]$$

$$= \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Therefore

$$\mathcal{L}\{e^{iat}\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Euler

Hence

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

and

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

Example 3.4

Compute $\mathcal{L}\{t^n\}$.

Solution

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st} dt = \left[\frac{-e^{-st}}{s} t^n\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= \left[\frac{-e^{-st}}{s} t^n\right]_0^\infty - \left[\frac{ne^{-st}}{s^2} t^{n-1}\right]_0^\infty + \frac{n(n-1)}{s^2}\int_0^\infty t^{n-2}e^{-st}dt$$

$$\therefore \mathcal{L}\{t^n\} - \frac{n(n-1)(n-2)\cdots}{s^n} \times \frac{1}{s} = \frac{n!}{s^{n+1}}$$

e.g $\mathcal{L}\{c\} = \frac{c}{s}$ and $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}\{10t^3\} = 10 \cdot \frac{3!}{s^{3+1}} = \frac{60}{s^4}$

Theorem 3.1

The Laplace operator is a linear operator. That it,

$$\mathcal{L}\left(af_1(t) + bf_2(t)\right) = a\mathcal{L}f_1(t) + b\mathcal{L}f_2(t)$$

Proof

$$\mathcal{L}\left(af_1(t) + bf_2(t)\right) = \int_0^\infty e^{-st}\left(af_1(t) + bf_2(t)\right) dt$$

$$= a\int_0^\infty e^{-st} f_1(t)dt + b\int_0^\infty e^{-st} f_2(t)dt = a\mathcal{L}f_1(t) + b\mathcal{L}f_2(t)$$

Theorem 3.2(Shift Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

Proof

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty e^{-(s-a)t}f(t)dt$$

Let $p = s - a$

$$\Rightarrow \int_0^\infty e^{-pt}f(t)dt = F(p) = F(s-a)$$

e.g $\mathcal{L}\{t^2e^{at}\}$ since $\mathcal{L}\{t^2\} = \dfrac{2!}{s^3} \Rightarrow \mathcal{L}\{t^2e^{at}\} = \dfrac{2!}{(s-a)^3}$.

Example 3.5

Find the Laplace transform of $f(t)$ define by

$$f(t) = \begin{cases} 5 & 0 < t < 3 \\ -2 & 3 < x < 5 \\ 0 & t \geq 5 \end{cases}$$

Solution

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt &= 5\int_0^3 e^{-st}dt - 2\int_3^5 e^{-st}dt + 0 \\
&= \left[\frac{5}{-s}e^{-st}\right]_0^3 + \left[\frac{2}{s}e^{-st}\right]_3^5 \\
&= \frac{5}{-s}e^{-3s} + \frac{5}{s} + \frac{2}{s}e^{-5s} - \frac{2}{s}e^{-3s} \\
&= \frac{5 - 7e^{-3s} + 2e^{-5s}}{s}
\end{aligned}$$

Example 3.6

Evaluate $\mathcal{L}(3t^5 + 10t^4e^t - 3t^3 + 5t^2 - t + 11)$.

Solution

Since Laplace transform is a linear operator we have

$$\begin{aligned}
&3\mathcal{L}\{t^5\} + 10\mathcal{L}\{t^4e^t\} - 3\mathcal{L}\{t^3\} + 5\mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{11\} \\
&= 3 \cdot \frac{5!}{s^6} + 10 \cdot \frac{4!}{(s-1)^5} - 3 \cdot \frac{3!}{s^4} + 5 \cdot \frac{2!}{s^3} - \frac{1}{s^2} + \frac{11}{s}.
\end{aligned}$$

Exercise 3.1

1. Find the Laplace transform of the following
   (a) $\mathcal{L}\{5e^{-6t}\}$
   (b) $\mathcal{L}\{6t^5 - 3t^2 - 10\}$
   (c) $\mathcal{L}\{6\sin 3t - 5\cos 3t\}$
2. Show that $\mathcal{L}\{\sinh at\} = \dfrac{s}{s^2 - a^2}$ and $\mathcal{L}\{\cosh at\} = \dfrac{a}{s^2 - a^2}$
3. Find the Laplace transform of the following functions
   (a) t^2e^{-3t}
   (b) $e^{-\alpha t}\cos\beta t$
   (c) $5e^{2t}\sinh 2t$
   (d) $2e^{-t}\cos^2\frac{1}{2}t$
   (e) $\sinh t\cos t$
   (f) $(t+1)^2e^t$

3.1.1 Existence of Laplace Transform

Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{kt} \qquad \forall t \geq 0 \tag{3.1}$$

and for some k and M then the Laplace transform of $f(t)$ exist for all $s > k$.

Proof

Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any interval on t-axis from (3.1) assuming that $s > k$ we obtain

$$|\mathcal{L}\{f(t)\}| = \left|\int_0^\infty e^{-st}f(t)dt\right| \leq \int_0^\infty |f(t)|e^{-st}dt$$

$$\leq \int_0^\infty Me^{kt}e^{-st}dt$$

$$\leq M\int_0^\infty e^{-(s-k)t}dt$$

$$\leq M \left[\frac{e^{-(s-k)t}}{-(s-k)} \right]_0^{\infty}$$

$$\leq \frac{M}{s-k}$$

where the condition $s > k$ was needed for the existence of the last integral. This complete the prove.

3.1.2 Inverse Laplace Transform

If the Laplace Transform o $f(t) = F(s)$ then the inverse transform of $F(s)$ denoted by $\mathcal{L}^{-1}[F(s)] = f(t)$ e.g If $\mathcal{L}\{t^2\} = \frac{2!}{s^3} \Rightarrow \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = t^2$

Example 3.7

Evaluate $\mathcal{L}^{-1}\left\{\frac{7s}{s^2+25}\right\}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{7s}{s^2+25}\right\} = 7\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} = 7\cos 5t$$

Example 3.8

Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{L^2s^2+n^2\pi^2}\right\}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{s}{L^2s^2+n^2\pi^2}\right\} = \frac{1}{L^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{n^2\pi^2}{L^2}}\right\} = \frac{1}{L^2}\cos\frac{n\pi}{L}t$$

Example 3.9

Evaluate $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-2s+5}\right\}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)+3}{(s-1)^2+4}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}$$

$$= 2\cdot e^t\cos 2t + \frac{3}{2}e^t\sin 2t.$$

Example 3.10

Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s-5)(s+3)}\right\}$

Solution

Using partial fraction $\frac{s}{(s-5)(s+3)} = \frac{A}{s-5} + \frac{B}{s+3}$ which give $A = \frac{5}{8}$ and $B = \frac{3}{8}$ then

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-5)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{8(s-5)} + \frac{3}{8(s+3)}\right\}$$

$$= \frac{5}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} + \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{5}{8}e^{5t} + \frac{3}{8}e^{-3t}$$

Exercise 3.2

Find the Laplace transform of the following functions

1. $\frac{0.1s+0.9}{s^2+3.24}$
2. $\frac{5s}{s^2-25}$
3. $\frac{-s-10}{s^2-s-2}$
4. $\frac{s-4}{s^2-4}$

5. $\frac{2.4}{s^4} - \frac{228}{s^8}$

6. $\frac{60 + 6s^2 + s^4}{s^7}$

7. $\frac{1 - 7s}{(s-3)(s-1)(s+2)}$

8. $\sum_{k=1}^{5} \frac{a_k}{s + k^2}$

9. $\frac{s^2 + 6s - 18}{s^5 - 3s^4}$

10. $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$

11. $\frac{2s^3}{s^4 - 1}$

12. $\frac{s}{(s+\frac{1}{2})^2 + 1}$

3.2 Laplace Transform of Derivatives

Theorem 3.3

If $f'(t)$ is piecewise continuous in $0 \leq t \leq T$ and of exponential order in $T \leq \infty$ then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof

$$\mathcal{L}\{f'(t)\} = \int_0^\infty s^{-st} f'(t)dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= s\mathcal{L}\{f(t)\} - f(0)$$

Theorem 3.4

If $f''(t)$ and $f'(t)$ are piecewise continuous in $0 \leq t \leq T$ then

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Solution

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \int_0^\infty e^{-st} f''(t)dt \\
&= \left[e^{-st} f'(t)\right]_0^\infty + s\int_0^\infty e^{-st} f'(t)dt \\
&= -f'(0) + s\left[\left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt\right] \\
&= -f'(0) + s\left[-f(0) + s\mathcal{L}\{f(t)\}\right] \\
&= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)
\end{aligned}$$

3.2.1 Application of Laplace transform in Solution of Differential Equations

Laplace transform can be use to solve a differential equations by taking the Laplace transform of the equation to obtain $\mathcal{L}\{y\}$ then taking inverse to obtain y.

Example 3.11

Solve the differential equation

$$y'' + 2y' + y = e^{-t},\ y(0) = -1,\ y'(0) = 1$$

Solution

Taking Laplace transform of both side we have

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$s^2\mathcal{L}\{y\} + s - 1 + 2s\mathcal{L}\{y\} + 2 + \mathcal{L}\{y\} = \frac{1}{s+1}$$

$$(s^2 + 2s + 1)\mathcal{L}\{y\} = \frac{1}{s+1} - (s+1)$$

$$(s+1)^2\mathcal{L}\{y\} = \frac{1}{s+1} - (s+1)$$

$$\mathcal{L}\{y\} = \frac{-1}{s+1} + \frac{1}{(s+1)^3}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= -e^{-t} + \frac{1}{2}t^2e^{-t}$$

$$y = \left(\frac{1}{2}t^2 - 1\right)e^{-t}$$

Example 3.12

Solve the equation

$$y'' - 3y' + 2y = 4, \; y(0) = 1, \; y'(0) = 0$$

Solution

Taking Laplace transform of both side we have

$$\mathcal{L}\{y'' + -3y' + 2y\} = \mathcal{L}\{4\}$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\}2\mathcal{L}\{y\} = \frac{4}{s}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 3[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{y\} = \frac{4}{s}$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} - s + 3 = \frac{4}{s}$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} = \frac{4}{s} + s - 3 = \frac{4 + s^2 - 3s}{s}$$

$$\mathcal{L}\{y\} = \frac{4 + s^2 - 3s}{s(s^2 - 3s + 2)} = \frac{s^2 - 3s + 4}{s(s-1)(s-2)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s^2-3s+4}{s(s-1)(s-2)}\right\}$$

Resolving into partial fraction

$$\frac{s^2-3s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$s^2-3s+4 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

put $s=0$

$$\Rightarrow 4 = 2A \Rightarrow A = 2$$

put $s=1$

$$\Rightarrow 2 = -B \Rightarrow B = -2$$

put $s=2$

$$\Rightarrow 2 = 2C \Rightarrow C = 1$$

$$\therefore \frac{s^2-3s+4}{s(s-1)(s-2)} = \frac{2}{s} - \frac{2}{s-1} + \frac{1}{s-2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y = 2 - 2e^t + e^{2t}$$

Exercise 3.3

Solve the following differential equations using Laplace transform.

1. $y'' + ay' - 2ay = 0,\ y(0) = 6,\ y'(0) = 0$
2. $y'' + 0.04y = 0.02t^2,\ y(0) = -25,\ y'(0) = 0$
3. $y'' + 2y' - 3y = 6e^{-2t},\ y(0) = 2,\ y'(0) = -14$
4. $y'' + y' = 2\cos t,\ y(0) = 3,\ y'(0) = 4$
5. $y' + 3y' = 10\sin t,\ y(0) = 0,$
6. $y'' - 4y' + 3y = 6t - 8,\ y(0) = 0,\ y'(0) = 0$
7. $y''' + 8y = 32t^3 - 16t,\ y''(0) = y'(0) = y(0) = 0$

3.3 Convolution

Another important property of the Laplace transform has to do with product of transforms. It often happens that we are given two transforms $F(s)$ and $G(s)$ whose inverses $f(t)$ and $g(t)$ we known and we would like to calculate the inverse of the product $H(s) = F(s)G(s)$ from those known inverses $f(t)$ and $g(t)$. This inverse $h(t)$ is written $(f * g)(t)$, which is a standard notation and is called the convolution of f and g. How can we find h from f and g? This is stated in the following theorem since the situation and task just described arise quite often in application. This theorem is of considerable practical important.

Theorem 3.5(convolution Theorem)

Let $f(t)$ and $g(t)$ satisfies the hypothesis of existence theorem then the product of their transforms $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ is the transform $H(s) = \mathcal{L}\{h(t)\}$ of the convolution $h(t)$ of $f(t)$ and $g(t)$, which is denoted by $(f * g)(t)$ and define by

$$h(t) = (f * g)(t) = \int_0^t f(r)g(t-r)dr.$$

Proof

Exercise.

Example 3.13

Using convolution find the inverse $h(t)$ of $H(s)$ if

$$H(s) = \frac{1}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1}$$

Solution

We know that each factor on the right has the inverse $\sin t$. Hence by convolution theorem we have

$$h(t) = \mathcal{L}^{-1}(h) = \sin t \cdot \sin t = \int_0^t \sin r \sin(t-r)dr.$$

$$= \frac{1}{2}\int_0^t -\cos t dr + \frac{1}{2}\int_0^t \cos(2r-t)dr$$

$$= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t.$$

Example 3.14

If the $\frac{1}{s^2}$ has the inverse t and $\frac{1}{s}$ has the inverse 1. use the convolution theorem to find the inverse of $\frac{1}{s^3} = \frac{1}{s^2}\cdot\frac{1}{s}$

Solution

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = t * 1 = \int_0^t t\cdot 1 d\tau = \left[\frac{\tau^2}{2}\right]_0^t = \frac{t^2}{2}$$

Example 3.15

Let $H(s) = \dfrac{1}{s^2(s-a)}$ find $h(t)$.

Solution

The inverse of $\frac{1}{s^2}$ is t and the inverse of $\frac{1}{s-a}$ is e^{at}. Using convolution theore. and integrating we have

$$h(t) = t * e^{at} = \int_0^t \tau e^{a(t-\tau)} d\tau = e^{at}\int_0^t \tau e^{-a\tau} d\tau$$

$$= e^{at}\left[\left(-\frac{\tau e^{-a\tau}}{a}\right)_0^t + \frac{1}{a}\int_0^t e^{-a\tau} d\tau\right]$$

$$= e^{at}\left[-\frac{\tau e^{-a\tau}}{a} - \frac{1}{a^2}e^{-a\tau}\right]_0^t$$

$$= e^{at}\left[-\frac{te^{-at}}{a} - \frac{1}{a^2}e^{-at} + \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}\left[e^{at} - at - 1\right]$$

Exercise 3.4

Find the inverse transform of the following by convolution.

1. $\dfrac{6}{s(s+3)}$
2. $\dfrac{1}{s(s^2+4)}$
3. $\dfrac{1}{(s+3)(s-2)}$
4. $\dfrac{1}{(s-a)^2}$

d exams

(4) in (1) and Convolution